Math 310, Lesieutre
Problem set #14
November 25, 2015

Problems for M 11/23:

6.5.9 *Find (a) the orthogonal projection of* $\mathbf{b}$ *onto* $\operatorname{Col} A$ *and (b) a least-squares solution to* $A\mathbf{x} = \mathbf{b}$.

$$A = \begin{bmatrix} 1 & 5 \\ 3 & 1 \\ -2 & 4 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 4 \\ -2 \\ -3 \end{bmatrix}.$$

Lucky for us the columns of A are already orthogonal. So we don't need to do any Gram–Schmidt, like we did for the example in lecture. Let $\mathbf{v}_1$ be the first column and $\mathbf{v}_2$ be the second column.

$$\begin{aligned} \operatorname{proj}_{\operatorname{Col} A} \mathbf{b} &= \frac{\mathbf{b} \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 + \frac{\mathbf{b} \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 \\ &= \frac{4}{14}\begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} + \frac{6}{42}\begin{bmatrix} 5 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}. \end{aligned}$$

Then we want to solve

$$A\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

Row reduction on the augmented matrix yields

$$\left[\begin{array}{cc|c} 1 & 5 & 1 \\ 3 & 1 & 1 \\ -2 & 4 & 0 \end{array}\right] \to \cdots \to \left[\begin{array}{cc|c} 1 & 0 & \frac{2}{7} \\ 0 & 1 & \frac{1}{7} \\ 0 & 0 & 0 \end{array}\right]$$

So the least-squares solution is

$$\hat{\mathbf{x}} = \begin{bmatrix} 2/7 \\ 1/7 \end{bmatrix}.$$

6.5.12 *Same deal, with*

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & -1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 2 \\ 5 \\ 6 \\ 6 \end{bmatrix}.$$

Again, we have the good fortune that the columns of this matrix are already orthogonal, so we can use the easy version of the formula for projection. Otherwise we would need to cook up an orthogonal basis using Gram–Schmidt. We have

$$\begin{aligned}\operatorname{proj}_{\operatorname{Col} A}\mathbf{b} &= \frac{\mathbf{b}\cdot\mathbf{v}_1}{\mathbf{v}_1\cdot\mathbf{v}_1}\mathbf{v}_1 + \frac{\mathbf{b}\cdot\mathbf{v}_2}{\mathbf{v}_2\cdot\mathbf{v}_2}\mathbf{v}_2 + \frac{\mathbf{b}\cdot\mathbf{v}_3}{\mathbf{v}_3\cdot\mathbf{v}_3}\mathbf{v}_3 \\ &= \frac{1}{3}\begin{bmatrix}1\\1\\0\\-1\end{bmatrix} + \frac{14}{3}\begin{bmatrix}1\\0\\1\\1\end{bmatrix} + \frac{-5}{3}\begin{bmatrix}0\\-1\\1\\-1\end{bmatrix} = \begin{bmatrix}5\\2\\3\\6\end{bmatrix}.\end{aligned}$$

Then we need to solve $A\mathbf{x} = \hat{\mathbf{b}}$:

$$\left[\begin{array}{ccc|c}1 & 1 & 0 & 5\\ 1 & 0 & -1 & 2\\ 0 & 1 & 1 & 3\\ -1 & 1 & -1 & 6\end{array}\right] \to \left[\begin{array}{ccc|c}1 & 0 & 0 & \frac{1}{3}\\ 0 & 1 & 0 & \frac{14}{3}\\ 0 & 0 & 1 & -\frac{5}{3}\\ 0 & 0 & 0 & 0\end{array}\right].$$

So the least-squares solution is

$$\hat{\mathbf{x}} = \begin{bmatrix}1/3\\14/3\\-5/3\end{bmatrix}.$$

6.6.2 *Find the equation of the least-squares line that best fits the points* $(1,0),(2,1),(4,2),(5,3)$.

Say our line is $y = mx + b$. Going through the point $(2,1)$ means that $2m + b = 1$, and similarly for the others. The linear system we want to solve is

$$\begin{bmatrix}1 & 1\\2 & 1\\4 & 1\\5 & 1\end{bmatrix}\begin{bmatrix}m\\b\end{bmatrix} = \begin{bmatrix}0\\1\\2\\3\end{bmatrix}.$$

To find the best fit, we want to instead solve $A^TA\mathbf{x} = A^T\mathbf{b}$, which is

$$\begin{aligned}\begin{bmatrix}1 & 2 & 4 & 5\\1 & 1 & 1 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\2 & 1\\4 & 1\\5 & 1\end{bmatrix}\begin{bmatrix}m\\b\end{bmatrix} &= \begin{bmatrix}1 & 2 & 4 & 5\\1 & 1 & 1 & 1\end{bmatrix}\begin{bmatrix}0\\1\\2\\3\end{bmatrix}\\ \begin{bmatrix}46 & 12\\12 & 4\end{bmatrix}\begin{bmatrix}m\\b\end{bmatrix} &= \begin{bmatrix}25\\6\end{bmatrix}\\ \begin{bmatrix}m\\b\end{bmatrix} &= \begin{bmatrix}7/10\\-3/5\end{bmatrix}.\end{aligned}$$

This means that the best-fit line is

$$y = \frac{7}{10}x - \frac{3}{5}.$$

6.6.3 *Find the equation of the least-squares line that best fits the points* $(-1, 0), (0, 1), (1, 2), (2, 4)$.

We do this one in exactly the same way. We want to solve

$$\begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 4 \end{bmatrix}.$$

No solutions, but the normal equations are instead given by

$$\begin{bmatrix} -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \\ 4 \end{bmatrix}$$
$$\begin{bmatrix} 6 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 10 \\ 7 \end{bmatrix}$$
$$\begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} \frac{13}{10} \\ \frac{11}{10} \end{bmatrix}.$$

So we want the line

$$y = \frac{13}{10}x + \frac{11}{10}.$$

Problems for W 11/25:

6.5.3 *Find a least-squares solution of* $A\mathbf{x} = \mathbf{b}$ *by (a) constructing the normal equations for* $\hat{\mathbf{x}}$ *and (b) solving for* $\hat{\mathbf{x}}$. *("Normal equations" is just the lingo for* $A^T A\mathbf{x} = A^T\mathbf{b}$*).*

$$A = \begin{bmatrix} 1 & -2 \\ -1 & 2 \\ 0 & 3 \\ 2 & 5 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 3 \\ 1 \\ -4 \\ 2 \end{bmatrix}.$$

The normal equations are given by

$$A^T A\mathbf{x} = A^T\mathbf{b}$$
$$\begin{bmatrix} 6 & 6 \\ 6 & 42 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 6 \\ -6 \end{bmatrix}.$$

The solution is then given by

$$\mathbf{x} = \begin{bmatrix} 6 & 6 \\ 6 & 42 \end{bmatrix}^{-1} \begin{bmatrix} 6 \\ -6 \end{bmatrix} = \frac{1}{36} \begin{bmatrix} 7 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ -6 \end{bmatrix} = \begin{bmatrix} \frac{4}{3} \\ -\frac{1}{3} \end{bmatrix}.$$

6.5.15 *Use the factorization $A = QR$ to find the least-squares solution of $A\mathbf{x} = \mathbf{b}$.*

$$A = \begin{bmatrix} 2 & 3 \\ 2 & 4 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2/3 & -1/3 \\ 2/3 & 2/3 \\ 1/3 & -2/3 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix}.$$

The formula to do this is $\hat{\mathbf{x}} = R^{-1}Q^T\mathbf{b}$, which in the situation in question gives

$$\begin{aligned} \hat{\mathbf{x}} = R^{-1}Q^T\mathbf{b} &= \begin{bmatrix} 3 & 5 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 2/3 & -1/3 \\ 2/3 & 2/3 \\ 1/3 & -2/3 \end{bmatrix}^T \begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix} \\ &= \frac{1}{3} \begin{bmatrix} 1 & -5 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2/3 & 2/3 & 1/3 \\ -1/3 & 2/3 & -2/3 \end{bmatrix} \begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix} \\ &= \frac{1}{9} \begin{bmatrix} 7 & -8 & 11 \\ -3 & 6 & -6 \end{bmatrix} \begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}. \end{aligned}$$

6.6.8 *A simple curve that often makes a good model for the variable costs of a company, as a function of the sales level x, has the form $y = \beta_1 x + \beta_2 x^2 + \beta_3 x^3$. There is no constant term because fixed costs are not included.*
Give the design matrix and the parameter vector for the linear model that leads to a least-squares fit of the equation above, with data $(x_1, y_1), \ldots, (x_n, y_n)$. (Your answer should be in terms of the x's and y's.)

We want to be able to plug in x_1 for x and have $\beta_1 x + \beta_2 x^2 + \beta_3 x^3$ spit out y_1. So our equation is going to be $\beta_1 x_1 + \beta_2 x_1^2 + \beta_3 x_1^3 = y_1$, and similarly for the other data points. In matrix form this is

$$\begin{bmatrix} x_1 & x_1^2 & x_1^3 \\ x_2 & x_2^2 & x_2^3 \\ & \ddots & \\ x_n & x_n^2 & x_n^3 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}.$$

6.6.9 *A certain experiment produces the data $(1, 7.9)$, $(2, 5.4)$, $(3, -0.9)$. Describe the model that produces a least-squares fit of these points by a function of the form $y = A\cos x + B\sin x$. (It'll be a mess to actually find A and B; just set up the matrices.)*

We want to be able to plug in $x = 1$ and get 7.9, etc. This gives $A\cos 1 + B\sin 1 = 7.9$, etc:

$$\begin{bmatrix} \cos 1 & \sin 1 \\ \cos 2 & \sin 2 \\ \cos 3 & \sin 3 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 7.9 \\ 5.4 \\ -0.9 \end{bmatrix}.$$

To find the best fit solution, we would work out the normal equations for the system and solve it. Since there isn't any way to simplify $\cos 1$ etc., we'd probably want to take a numerical approximation for all of the trig functions involved.