Math 310, Lesieutre
Problem set #6
October 7, 2015

Problems for M 9/28:

3.1.1 *Compute the determinant of the following matrix by using cofactor expansion across the first row. Also compute it using cofactor expansion down the second column. (You should get the same answer either way.)*

$$\begin{vmatrix} 3 & 0 & 4 \\ 2 & 3 & 2 \\ 0 & 5 & -1 \end{vmatrix}.$$

Using the first row, we get

$$3\begin{vmatrix} 3 & 2 \\ 5 & -1 \end{vmatrix} + 4\begin{vmatrix} 2 & 3 \\ 0 & 5 \end{vmatrix} = 3(-13) + 4(10) = 1.$$

Using the second column, we have

$$3\begin{vmatrix} 3 & 4 \\ 0 & -1 \end{vmatrix} + -5\begin{vmatrix} 3 & 4 \\ 2 & 2 \end{vmatrix} = 3(-3) - 5(6-8) = 1.$$

3.1.4 *Same deal, for*

$$\begin{vmatrix} 1 & 2 & 4 \\ 3 & 1 & 1 \\ 2 & 4 & 2 \end{vmatrix}$$

Across the first row we get

$$1\begin{vmatrix} 1 & 1 \\ 4 & 2 \end{vmatrix} - 2\begin{vmatrix} 3 & 1 \\ 2 & 2 \end{vmatrix} + 4\begin{vmatrix} 3 & 1 \\ 2 & 4 \end{vmatrix} = 1(-2) - 2(4) + 4(10) = 30.$$

Now the first column we get

$$1\begin{vmatrix} 1 & 1 \\ 4 & 2 \end{vmatrix} - 3\begin{vmatrix} 2 & 4 \\ 4 & 2 \end{vmatrix} + 2\begin{vmatrix} 2 & 4 \\ 1 & 1 \end{vmatrix} = 1(-2) - 3(-12) + 2(-2) = 30.$$

3.1.9 *Compute the determinant by cofactor expansion. At each step, choose a row or column that involves the least amount of computation. (You'll want to pick rows or columns with lots of zeroes if you can.)*

$$\begin{vmatrix} 4 & 0 & 0 & 5 \\ 1 & 7 & 2 & -5 \\ 3 & 0 & 0 & 0 \\ 8 & 3 & 1 & 7 \end{vmatrix}$$

First we use the third row:

$$\begin{vmatrix} 4 & 0 & 0 & 5 \\ 1 & 7 & 2 & -5 \\ 3 & 0 & 0 & 0 \\ 8 & 3 & 1 & 7 \end{vmatrix} = 3 \begin{vmatrix} 0 & 0 & 5 \\ 7 & 2 & -5 \\ 3 & 1 & 7 \end{vmatrix}$$

Now the third row:

$$3 \begin{vmatrix} 0 & 0 & 5 \\ 7 & 2 & -5 \\ 3 & 1 & 7 \end{vmatrix} = 3 \cdot 5 \begin{vmatrix} 7 & 2 \\ 3 & 1 \end{vmatrix} = 15.$$

3.1.15 *Compute the following 3×3 determinant by adding up the products of the downward diagonals and subtracting the products of the upward diagonals.*

$$\begin{vmatrix} 1 & 0 & 4 \\ 2 & 3 & 2 \\ 0 & 5 & -2 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 0 & 4 \\ 2 & 3 & 2 \\ 0 & 5 & -2 \end{vmatrix} = (1)(3)(-2)+(0)(2)(0)+(4)(2)(5)-(0)(3)(4)-(2)(0)(-2)-(1)(5)(2) = 24.$$

3.1.19 *Compute the determinants of the following two matrices. Say which row operation relates them, and deduce how performing that row operation changes the determinant.*

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

The determinants are $ad - bc$ and $bc - ad = -(ad - bc)$. They differ by a row swap, and we can see that a row swap changes the determinant by a factor of -1.

Problems for W 9/30:

None! That day was the midterm. You can try some extra practice determinants if you are looking for something to do – the more the better.

Problems for F 10/2:

3.2.5 *Find the determinant using row reduction to echelon form.*

$$\begin{vmatrix} 1 & 5 & -4 \\ -1 & -4 & 5 \\ -2 & -8 & 7 \end{vmatrix}.$$

You're all good at row reduction now.

$$\begin{vmatrix} 1 & 5 & -4 \\ -1 & -4 & 5 \\ -2 & -8 & 7 \end{vmatrix} \to \begin{vmatrix} 1 & 5 & -4 \\ 0 & 1 & 1 \\ -2 & -8 & 7 \end{vmatrix} \to \begin{vmatrix} 1 & 5 & -4 \\ 0 & 1 & 1 \\ 0 & 2 & -1 \end{vmatrix} \to \begin{vmatrix} 1 & 5 & -4 \\ 0 & 1 & 1 \\ 0 & 0 & -3 \end{vmatrix}$$

This means that the determinant is $(1)(1)(-3) = -3$.

3.2.11 *Combine the methods of row reduction and cofactor expansion to compute the following determinant.*

$$\begin{vmatrix} 3 & 4 & -3 & -1 \\ 3 & 0 & 1 & -3 \\ -6 & 0 & -4 & 3 \\ 6 & 8 & -4 & -1 \end{vmatrix}.$$

There are a few ways to do this; mine is probably not the fastest. Let's expand down the second column.

$$\begin{vmatrix} 3 & 4 & -3 & -1 \\ 3 & 0 & 1 & -3 \\ -6 & 0 & -4 & 3 \\ 6 & 8 & -4 & -1 \end{vmatrix} = -4 \begin{vmatrix} 3 & 1 & -3 \\ -6 & -4 & 3 \\ 6 & -4 & -1 \end{vmatrix} + 8 \begin{vmatrix} 3 & -3 & -1 \\ 3 & 1 & -3 \\ -6 & -4 & 3 \end{vmatrix}$$

But

$$\begin{vmatrix} 3 & 1 & -3 \\ -6 & -4 & 3 \\ 6 & -4 & -1 \end{vmatrix} \to \begin{vmatrix} 3 & 1 & -3 \\ 0 & -2 & -3 \\ 6 & -4 & -1 \end{vmatrix} \to \begin{vmatrix} 3 & 1 & -3 \\ 0 & -2 & -3 \\ 0 & -6 & 5 \end{vmatrix} \to \begin{vmatrix} 3 & 1 & -3 \\ 0 & -2 & -3 \\ 0 & 0 & 14 \end{vmatrix}.$$

So the first 3×3 determinant is $(3)(-2)(14) = -84$. The second one is

$$\begin{vmatrix} 3 & -3 & -1 \\ 3 & 1 & -3 \\ -6 & -4 & 3 \end{vmatrix} \to \begin{vmatrix} 3 & -3 & -1 \\ 0 & 4 & -2 \\ -6 & -4 & 3 \end{vmatrix} \to \begin{vmatrix} 3 & -3 & -1 \\ 0 & 4 & -2 \\ 0 & -10 & 1 \end{vmatrix} \to \begin{vmatrix} 3 & -3 & -1 \\ 0 & 4 & -2 \\ 0 & 0 & -4 \end{vmatrix}$$

The second one is $(3)(4)(-4) = -48$. So the big determinant is

$$-4(-84) + 8(-48) = 336 - 384 = -48.$$

3.2.24 *Use determinants to decide if these vectors are linearly independent:*

$$\begin{bmatrix} 4 \\ 6 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} -7 \\ 0 \\ 7 \end{bmatrix}, \quad \begin{bmatrix} -3 \\ -5 \\ -2 \end{bmatrix}.$$

The columns of a matrix are linearly independent if the matrix is invertible, and it's invertible if the determinant is nonzero. So we just need to stick these three vectors in as the columns of a matrix and compute the determinant; if it's 0 they're dependent, and if it's not, they're independent.

$$A = \begin{bmatrix} 4 & -7 & -3 \\ 6 & 0 & -5 \\ 2 & 7 & -2 \end{bmatrix}$$

$$\det A = -(-7) \begin{bmatrix} 6 & -5 \\ 2 & -2 \end{bmatrix} - 7 \begin{bmatrix} 4 & -3 \\ 6 & -5 \end{bmatrix} = 7(-2) - 7(-2) = 0.$$

So the vectors are dependent.

3.2.29 *Compute* $\det B^4$, *where*

$$B = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}.$$

We have

$$\det B = (1)(1)(1) + (0)(2)(1) + (1)(1)(2) - (1)(1)(1) - (0)(1)(1) - (2)(2)(1) = -2.$$

This means

$$\det B^4 = (\det B)^4 = (-2)^4 = 16.$$

Luckily we don't have to actually compute B^4.