Math 310, Lesieutre
Problem set #7
October 14, 2015

Problems for M 10/5:

3.3.1 *Use Cramer's rule to solve* $5x_1 + 7x_2 = 3$, $2x_1 + 4x_2 = 1$.

We have

$$A = \begin{bmatrix} 5 & 7 \\ 2 & 4 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

Note $\det A = 6$. According to Cramer's rule,

$$x_1 = \frac{\det A_1(\mathbf{b})}{\det A} = \frac{\begin{vmatrix} 3 & 7 \\ 1 & 4 \end{vmatrix}}{6} = \frac{5}{6}.$$

$$x_2 = \frac{\det A_2(\mathbf{b})}{\det A} = \frac{\begin{vmatrix} 5 & 3 \\ 2 & 1 \end{vmatrix}}{6} = -\frac{1}{6}.$$

3.3.7 *Determine the values of* s *for which the system has a unique solution, and describe the solution:*

$$\begin{aligned} 6sx_1 + 4x_2 &= 5 \\ 9x_1 + 2sx_2 &= -2. \end{aligned}$$

This is another one for Cramer's rule. Use

$$A = \begin{bmatrix} 6s & 4 \\ 9 & 2s \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}.$$

This time $\det A = 12s^2 - 36$.

$$x_1 = \frac{\det A_1(\mathbf{b})}{\det A} = \frac{\begin{vmatrix} 5 & 4 \\ -2 & 2s \end{vmatrix}}{12s^2 - 36} = \frac{10s + 8}{12s^2 - 36}.$$

$$x_2 = \frac{\det A_2(\mathbf{b})}{\det A} = \frac{\begin{vmatrix} 6s & 5 \\ 9 & -2 \end{vmatrix}}{12s^2 - 36} = -\frac{12s + 45}{12s^2 - 36}.$$

There's a unique solution as long as $12s^2 - 36$ is not 0, which is to say that $s \neq \pm\sqrt{3}$. The solution then is what it says above.

3.3.11 *Find the adjugate of the given matrix and use it to find the inverse:*

$$A = \begin{bmatrix} 0 & -2 & -1 \\ 5 & 0 & 0 \\ -1 & 1 & 1 \end{bmatrix}$$

(We didn't define the adjugate in class, though we got close: take a look at page 181.)

The minors we need are

$$\begin{vmatrix} 0 & 0 \\ 1 & 1 \end{vmatrix} = 0 \qquad \begin{vmatrix} 5 & 0 \\ -1 & 1 \end{vmatrix} = 5 \qquad \begin{vmatrix} 5 & 0 \\ -1 & 1 \end{vmatrix} = 5$$
$$\begin{vmatrix} -2 & -1 \\ 1 & 1 \end{vmatrix} = -1 \qquad \begin{vmatrix} 0 & -1 \\ -1 & 1 \end{vmatrix} = -1 \qquad \begin{vmatrix} 0 & -2 \\ -1 & 1 \end{vmatrix} = -2$$
$$\begin{vmatrix} -2 & -1 \\ 0 & 0 \end{vmatrix} = 0 \qquad \begin{vmatrix} 0 & -1 \\ 5 & 0 \end{vmatrix} = 5 \qquad \begin{vmatrix} 0 & -2 \\ 5 & 0 \end{vmatrix} = 10$$

To get the adjugate, with stick these guys in a grid with the usual alternating + and − signs, then take the transpose.

$$\operatorname{adj}(A) = \begin{bmatrix} 0 & 1 & 0 \\ -5 & -1 & -5 \\ 5 & 2 & 10 \end{bmatrix}$$

To find the inverse, we need to know $\det A$. Expanding by cofactors in the second row, that's

$$-5\begin{vmatrix} -2 & -1 \\ 1 & 1 \end{vmatrix} = (-5)(-1) = 5.$$

So

$$A^{-1} = \begin{bmatrix} 0 & \frac{1}{5} & 0 \\ -1 & -\frac{1}{5} & -1 \\ 1 & \frac{2}{5} & 2 \end{bmatrix}.$$

3.3.20 *Find the area of the parallelogram with vertices* $(0, 0)$, $(2, -4)$, $(4, -5)$, *and* $(2, -1)$

Remember that determinant gives the area of a parallelogram (taking absolute value, anyway). The area we're after is

$$\text{area} = \begin{vmatrix} 2 & 2 \\ -4 & -1 \end{vmatrix} = 6.$$

3.3.29 *Find a formula for the area of the triangle whose vertices are* $\mathbf{0}$, $\mathbf{v}_1 = (a, b)$, *and* $\mathbf{v}_2 = (c, d)$ *in* $\mathbb{R}^2$.

The triangle is half of the parallelogram with legs those vectors, so

$$\text{area} = \frac{1}{2}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{ad - bc}{2}.$$

Problems for W 10/7:

4.1.1 *Let V be the first quadrant in the xy-plane, that is the set of all vectors (x, y) with $x \geq 0$ and $y \geq 0$. In set notation, this is:*

$$V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x \geq 0, y \geq 0 \right\}.$$

(a) *If $\mathbf{u}$ and $\mathbf{v}$ are in V, is $\mathbf{u} + \mathbf{v}$ in V?*

Yes, it is: if you add two vectors where both of the entries are positive, the sum has both entries positive.

(b) *Find a specific vector $\mathbf{u}$ in V and a specific scalar c such that $c\mathbf{u}$ is not in V. (This is enough to show that V is not a vector space.)*

Take

$$\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

and $c = -1$. Then

$$c\mathbf{v} = \begin{bmatrix} -1 \\ -1 \end{bmatrix},$$

which isn't in V. But this means that V isn't a subspace.

4.1.3 *Let H be the set of points inside and on the unit circle in the xy-plane:*

$$H = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x^2 + y^2 \leq 1 \right\}.$$

Find a specific example – two vectors or a vector and a scalar – to show that H is not a subspace of $\mathbb{R}^2$.

Take

$$\mathbf{v} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}.$$

Then $\mathbf{v}$ is in V because $(1/2)^2 + (1/2)^2 = 1/2$. Set $c = 2$. Then

$$c\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

which isn't in V because $1^2 + 1^2 = 2 > 1$. This means that V isn't a subspace.

4.1.5 *Let $\mathbb{P}_2$ be the vector space of polynomials of degree at most 2. Is the set of polynomials of the form at^2 a subset of $\mathbb{P}_2$ (where a is a scalar?)*

It is a subspace. We have $(at^2) + (bt^2) = (a + b)t^2$, which is also of the required form. So the set is closed under addition. Also, $c(at^2) = (ac)t^2$, as needed.

4.1.7 *Let $\mathbb{P}_2$ be the vector space of polynomials of degree at most 3. Is the set of all polynomials with integers as coefficients a subspace?*

Nope this one isn't a subspace. Take the vector $\mathbf{v} = t^3$, which has integer coefficients. Take $c = 1/2$. Then $c\mathbf{v} = t^3/2$, which doesn't have integers as coefficients.

Problems for F 10/9:

4.1.11 *Let W be the set of all vectors of the form* $\begin{bmatrix} 5b+2c \\ b \\ c \end{bmatrix}$. *Find vectors* $\mathbf{u}$ *and* $\mathbf{v}$ *such that* $W = \operatorname{span}(\mathbf{u}, \mathbf{v})$. *Why does this show that W is a subspace of $\mathbb{R}^n$?*

This is the span of

$$\mathbf{u} = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix},$$

since

$$b\mathbf{u} + c\mathbf{v} = b\begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 5b+2c \\ b \\ c \end{bmatrix}.$$

It shows that W is a subspace because the span of any set of vectors is always a subspace.

4.2.4 *Find an explicit description of the nullspace of the following matrix by listing a set of vectors that span the nullspace:*

$$A = \begin{bmatrix} 1 & -6 & 4 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

We want to find the general solution in parametric vector form, and then take the vectors that show up here. These span the set (in fact, they're a basis for the nullspace in the lingo from this week).

We need to get to rref:

$$\begin{bmatrix} 1 & -6 & 4 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -6 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -6 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

The variables x_2 and x_4 are free, and $x_1 = 6x_2$ and $x_3 = 0$. The parametric vector form is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = s\begin{bmatrix} 6 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

A basis for the nullspace is

$$\mathbf{u} = \begin{bmatrix} 6 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

4.2.7 *Explain why the following set either is or is not a vector space:*

$$\left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} : a + b + c = 2 \right\}$$

This is not a vector space. For example, $\mathbf{v} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$ is in the space, since $2 + 0 + 0 = 2$. But if we multiply by the scalar 7, we get $7\mathbf{v} = \begin{bmatrix} 14 \\ 0 \\ 0 \end{bmatrix}$, which isn't, since $14 + 0 + 0 \neq 2$. So the set in question is not closed under multiplication by scalars (in fact it isn't closed under addition, either.)

4.2.15 *Find a matrix A such that the given set is the column space of A.*

$$\left\{ \begin{bmatrix} 2s + 3t \\ r + s - 2t \\ 4r + s \\ 3r - s - t \end{bmatrix} : r, s, t \textit{ are scalars} \right\}.$$

This is a lot like the one we just did.

$$A = \begin{bmatrix} 0 & 2 & 3 \\ 1 & 1 & -2 \\ 4 & 1 & 0 \\ 3 & -1 & -1 \end{bmatrix}.$$

4.2.17 *For what value of k is* $\mathrm{Nul}(A)$ *a subspace of* $\mathbb{R}^k$*? For what value of k is* $\mathrm{Col}(A)$ *a subspace of* $\mathbb{R}^k$*?*

$$A = \begin{bmatrix} 2 & -6 \\ -1 & 3 \\ -4 & 12 \\ 3 & -9 \end{bmatrix}.$$

Remember that for a $m \times n$ matrix, the nullspace is a subspace of $\mathbb{R}^n$ (the number of columns), while the column space is a subspace of $\mathbb{R}^m$ (the number of rows). This matrix is 4×2, so the nullspace is a subspace of $\mathbb{R}^2$, and the column space is a subspace of $\mathbb{R}^4$. (Once you get to the bottom of things, this problem is really just asking you what the dimensions of the matrix are.)