Math 310, Lesieutre
Problem set #8
October 21, 2015

Problems for M 10/12:

4.3.3 *Determine whether these vectors are a basis for $\mathbb{R}^3$ by checking whether the vectors span $\mathbb{R}^3$, and whether the vectors are linearly independent.*

$$\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix}, \quad \begin{bmatrix} -3 \\ -5 \\ 1 \end{bmatrix}.$$

First let's check if they span. Let A be the matrix with these three vectors as the columns. That they span means that any $\mathbf{b}$ is a combination of these three vectors, or in other words, that $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$. So we want to check if the transformation given by A is onto. To check that is a matter of row reduction.

$$\begin{bmatrix} 1 & 3 & -3 \\ 0 & 2 & -5 \\ -2 & -4 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & -3 \\ 0 & 2 & -5 \\ 0 & 2 & -5 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & -3 \\ 0 & 2 & -5 \\ 0 & 0 & 0 \end{bmatrix}.$$

Uhoh! We got a row of zeroes, and there's no pivot in the third row. So these vectors don't span $\mathbb{R}^3$. (In fact, a basis for the span is given by the pivot columns of A, i.e. the first two vectors on the list.)

They're not linearly independent, either, since there's no pivot in the third column: the third vector is a linear combination of the other two.

4.3.9 *Find a basis for the nullspace of the following matrix:*

$$\begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 1 & -5 & 4 \\ 3 & -2 & 1 & -2 \end{bmatrix}.$$

We want to solve $A\mathbf{x} = \mathbf{0}$. Using the augmented matrix:

$$\left[\begin{array}{cccc|c} 1 & 0 & -3 & 2 & 0 \\ 0 & 1 & -5 & 4 & 0 \\ 3 & -2 & 1 & -2 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & -3 & 2 & 0 \\ 0 & 1 & -5 & 4 & 0 \\ 0 & -2 & 10 & -8 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & -3 & 2 & 0 \\ 0 & 1 & -5 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

So the variables x_3 and x_4 are free, and the general solution is:

$$\begin{cases} x_1 = 3s - 2t \\ x_2 = 5s - 4t \\ x_3 = s \\ x_4 = t \end{cases}$$

In parametric vector form,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = s \begin{bmatrix} 3 \\ 5 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ -4 \\ 0 \\ 1 \end{bmatrix}$$

A basis for the nullspace of A is therefore given by

$$\begin{bmatrix} 3 \\ 5 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -2 \\ -4 \\ 0 \\ 1 \end{bmatrix}.$$

4.3.15 *Find a basis for the space spanned by the vectors below:*

$$\begin{bmatrix} 1 \\ 0 \\ -3 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 2 \\ -3 \end{bmatrix}, \quad \begin{bmatrix} -3 \\ -4 \\ 1 \\ 6 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -3 \\ -8 \\ 7 \end{bmatrix} \quad \begin{bmatrix} 2 \\ 1 \\ -6 \\ 9 \end{bmatrix}.$$

This amounts to the same thing as finding the column space of the matrix whose columns are these vectors. And that's something you can figure out using row reduction: the column space is spanned by the pivot columns. I hope you will forgive me for not typing it all out. Here's what I got:

$$\begin{bmatrix} 1 & 0 & -3 & 1 & 2 \\ 0 & 1 & -4 & -3 & 1 \\ -3 & 2 & 1 & -8 & -6 \\ 2 & -3 & 6 & 7 & 9 \end{bmatrix} \xrightarrow{\text{blah blah blah}} \begin{bmatrix} 1 & 0 & -3 & 0 & 4 \\ 0 & 1 & -4 & 0 & -5 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The pivot columns are 1, 2, and 4, which means that the basis we want is vectors 1, 2, and 4 (from the original matrix):

$$\begin{bmatrix} 1 \\ 0 \\ -3 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} -3 \\ -4 \\ 1 \\ 6 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -3 \\ -8 \\ 7 \end{bmatrix}.$$

4.3.34 *Consider the polynomials* $\mathbf{p}_1(t) = 1 + t$, $\mathbf{p}_2(t) = 1 - t$, *and* $\mathbf{p}_3(t) = 2$. *Write down a linear dependence among these three polynomials. Find a basis for the span of these three polynomials.*

From looking at it, you can tell that $\mathbf{p}_1(t) + \mathbf{p}_2(t) - \mathbf{p}_3(t) = 0$; that's the linear dependence we're after. A basis for the span is then given by the first two vectors $\mathbf{p}_1(t)$ and $\mathbf{p}_2(t)$.

(If you want to do this without just guessing the answer, use the method of 4.4.14 below: the trick is to find a linear dependence between the coordinate vectors corresponding to these polynomials.)

Problems for W 10/14:

4.4.3 *Find the vector* $\mathbf{x}$ *determined by the given coordinate vector* $[\mathbf{x}]_{\mathcal{B}}$ *and the given basis* $\mathcal{B}$*:*

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 4 \\ -7 \\ 0 \end{bmatrix} \right\}, \qquad [\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}.$$

This is

$$3 \begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix} - 1 \begin{bmatrix} 4 \\ -7 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -5 \\ 9 \end{bmatrix}$$

4.4.5 *Find the coordinate vector* $[\mathbf{x}]_{\mathcal{B}}$ *of* $\mathbf{x}$ *relative to the given basis* $\mathcal{B}$*:*

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ -5 \end{bmatrix} \right\}, \qquad \mathbf{x} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

The low-thought approach is just to use the formula $[\mathbf{x}]_{\mathcal{B}} = \mathcal{P}_{\mathcal{B}}^{-1}\mathbf{x}$:

$$\begin{aligned} \mathcal{P} &= \begin{bmatrix} 1 & 2 \\ -3 & -5 \end{bmatrix} \\ \mathcal{P}^{-1} &= \frac{1}{(-5)-(-6)} \begin{bmatrix} -5 & -2 \\ 3 & 1 \end{bmatrix} \\ [\mathbf{x}]_{\mathcal{B}} = \mathcal{P}^{-1}\mathbf{x} &= \begin{bmatrix} -5 & -2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ -5 \end{bmatrix} \end{aligned}$$

4.4.7

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \\ 9 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix} \right\}, \qquad \mathbf{x} = \begin{bmatrix} 8 \\ -9 \\ 6 \end{bmatrix}.$$

As in the last problem, we want $\mathcal{P}_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = \mathbf{x}$. It's 3×3, so it will be quicker to use the augmented matrix and row reduce than to compute the inverse (which is a hassle):

$$\left[\begin{array}{ccc|c} 1 & -3 & 2 & 8 \\ -1 & 4 & -2 & -9 \\ -3 & 9 & 4 & 6 \end{array}\right] \xrightarrow{\cdots} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right],$$

which means that

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix}.$$

4.4.12 *Use an inverse matrix to find* $[\mathbf{x}]_{\mathcal{B}}$ *for the given* $\mathbf{x}$ *and* $\mathcal{B}$.

$$\mathcal{B} = \left\{ \begin{bmatrix} 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 6 \\ 7 \end{bmatrix} \right\}, \qquad \mathbf{x} = \begin{bmatrix} 2 \\ -6 \end{bmatrix}.$$

I did one of the earlier ones like this.

$$\begin{aligned} \mathcal{P} &= \begin{bmatrix} 4 & 6 \\ 5 & 7 \end{bmatrix} \\ \mathcal{P}^{-1} &= \frac{1}{28-30} \begin{bmatrix} 7 & -6 \\ -5 & 4 \end{bmatrix} \\ [\mathbf{x}]_{\mathcal{B}} = \mathcal{P}^{-1}\mathbf{x} &= \begin{bmatrix} -5 & -2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ -5 \end{bmatrix} \end{aligned}$$

4.4.14 *The set* $\mathcal{B} = \{1 - t^2, t - t^2, 2 - 2t + t^2\}$ *is a basis for* $\mathbb{P}_2$. *Find the coordinate vector of* $\mathbf{p}(t) = 3 + t - 6t^2$. *relative to* $\mathcal{B}$.

We can use the usual basis $1, t, t^2$ for $\mathbb{P}_2$, and translate this into a problem about normal vectors. We want to write

$$\begin{bmatrix} 3 \\ 1 \\ -6 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

This is done by row reduction on the augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & -2 & 1 \\ -1 & -1 & 1 & -6 \end{array}\right] \overset{\cdots}{\rightarrow} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -2 \end{array}\right].$$

That's the coordinate vector:

$$[\mathbf{p}(t)]_{\mathcal{B}} = \begin{bmatrix} 7 \\ -3 \\ -2 \end{bmatrix}.$$

Let's double-check that it actually works:

$$7(1 - t^2) - 3(t - t^2) - 2(2 - 2t + t^2) = 7 - 7t^2 - 3t + 3t^2 - 4 + 4t - 2t^2 = 3 + t - 6t^2,$$

as desired.

Problems for F 10/16:

4.5.1 *For the following subspace, find a basis, and state the dimension:*

$$\left\{ \begin{bmatrix} s - 2t \\ s + t \\ 3t \end{bmatrix} : s, t \in \mathbb{R} \right\}.$$

This is the column space of

$$A = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 0 & 3 \end{bmatrix}$$

The two columns are not parallel, so these two vectors are linearly independent. That means they are a basis for the column space. Since there are two basis vectors, this space is 2-dimensional.

4.5.5

$$\left\{ \begin{bmatrix} a - 4b - 2c \\ 2a + 5b - 4c \\ -a + 2c \\ -3a + 7b + 6c \end{bmatrix} : s, t \in \mathbb{R} \right\}.$$

This is the column space of

$$A = \begin{bmatrix} 1 & -4 & -2 \\ 2 & 5 & -4 \\ -1 & 0 & 2 \\ -3 & 7 & 6 \end{bmatrix}.$$

Rref for this matrix is

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

There is no pivot in the third column, but there are in columns 1 and 2. The first two vectors give a basis.

4.5.11 *Find the dimension of the subspace spanned by the following vectors:*

$$\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 9 \\ 4 \\ -2 \end{bmatrix}, \begin{bmatrix} -7 \\ -3 \\ 1 \end{bmatrix}.$$

Same deal; this is the column space of

$$\begin{bmatrix} 1 & 3 & 9 & -7 \\ 0 & 1 & 4 & -3 \\ 2 & 1 & -2 & 1 \end{bmatrix}$$

Rref for this guy is

$$\begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The first two columns are the pivot columns, and these give a basis for the column space. Since the basis has two vectors, the dimension of the subspace these things span is 2.

4.5.21 *The first four Hermite polynomials are* 1, $2t$, $2-4t+t^2$, *and* $6-18t+9t^2-t^3$. *Show that these polynomials form a basis for* $\mathbb{P}_3$.

We can check this using coordinates in $\mathbb{P}_3$ with respect to the basis 1, t, t^2, and t^3. To do that, we need to check that the corresponding vectors

$$\begin{bmatrix} 1 & 0 & 2 & 6 \\ 0 & 2 & -4 & -18 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

give a basis for $\mathbb{R}^4$. Do they span? Yes: the matrix is already in echelon form, and we can see that there's a pivot in every row. Are they independent? Yes: there is a pivot in every column. So these vectors are a basis for $\mathbb{R}^4$, which means that the polynomials 1, $2t$, $2-4t+t^2$, $6-18t+9t^2-t^3$ are a basis for $\mathbb{P}_3$.