Math 310, Lesieutre
Problem set #9
October 28, 2015

Problems for M 10/19:

4.6.1 *Row reduction on the matrix A below yields the matrix B. Without calculations, list* $\operatorname{rank} A$ *and* $\dim \operatorname{Nul} A$. *Find bases for* $\operatorname{Col} A$, $\operatorname{Row} A$, *and* $\operatorname{Nul} A$.

$$A = \begin{bmatrix} 1 & -4 & 9 & -7 \\ -1 & 2 & -4 & 1 \\ 5 & -6 & 10 & 7 \end{bmatrix}$$
$$B = \begin{bmatrix} 1 & 0 & -1 & 5 \\ 0 & -2 & 5 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We can see from the echelon form B that there are two pivot variables, x_1 and x_2. This means the dimension of the column space is 2, so the rank is 2. The dimension of the nullspace is $4 - 2 = 2$ (corresponding to the two free variables).

A basis for $\operatorname{Col} A$ is given by the two pivot columns, from the original matrix A:

$$\begin{bmatrix} 1 \\ -1 \\ 5 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \\ -6 \end{bmatrix}.$$

A basis for $\operatorname{Row} A$ is given by the nonzero rows of the matrix B:

$$\begin{bmatrix} 1 \\ 0 \\ -1 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 5 \\ -6 \end{bmatrix}.$$

To find the nullspace, we want the general solution to $A\mathbf{x} = \mathbf{0}$ in parametric vector form. Rref for B is

$$B = \begin{bmatrix} 1 & 0 & -1 & 5 \\ 0 & 1 & -5/2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This gives

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} s - 5t \\ 5/2s - 3t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} 1 \\ 5/2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -5 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

So a basis for the nullspace is

$$\begin{bmatrix} 2 \\ 5 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

(I multiplied the first vector by 2 to get rid of fractions; this doesn't change the fact that the vectors give a basis.)

4.6.2 *Same deal.*

$$A = \begin{bmatrix} 1 & -3 & 4 & -1 & 9 \\ -2 & 6 & -6 & -1 & -10 \\ -3 & 9 & -6 & -6 & -3 \\ 3 & -9 & 4 & 9 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & -3 & 0 & 5 & -7 \\ 0 & 0 & 2 & -3 & 8 \\ 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The solution is similar too. The variables x_1, x_3, and x_5 are pivots, and the other two are free. The dimension of the column space is 3 (also known as the rank), and the dimension of the nullspace is 2. A basis for the column space is given by the pivot columns:

$$\begin{bmatrix} 1 \\ -2 \\ -3 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ -6 \\ -6 \\ 4 \end{bmatrix}, \begin{bmatrix} 9 \\ -10 \\ -3 \\ 0 \end{bmatrix}.$$

A basis for the row space is the nonzero rows of B:

$$\begin{bmatrix} 1 \\ -3 \\ 0 \\ 5 \\ -7 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \\ -3 \\ 8 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 5 \end{bmatrix}.$$

To find the nullspace, we do some row reduction and get to rref:

$$R = \begin{bmatrix} 1 & -3 & 0 & 5 & 0 \\ 0 & 0 & 1 & -3/2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 3s - 5t \\ s \\ 3/2t \\ t \\ 0 \end{bmatrix} = s \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -5 \\ 0 \\ 3/2 \\ 1 \\ 0 \end{bmatrix}.$$

A basis for the nullspace is

$$\begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 3/2 \\ 1 \\ 0 \end{bmatrix}.$$

4.6.8 *Suppose that a 5×6 matrix A has four pivot columns. What is* $\dim \operatorname{Nul} A$*? Is* $\operatorname{Col} A = \mathbb{R}^4$*? Why or why not?*

Four pivot columns tells us that the rank is 4. We know $\operatorname{rank} A + \dim \operatorname{Nul} A = 6$, so $\dim \operatorname{Nul} A = 2$. It's not quite right to say that $\operatorname{Col} A = \mathbb{R}^4$. $\operatorname{Col} A$ is a 4-dimensional subspace of $\mathbb{R}^5$.

4.6.9 *If the null space of a 5×6 matrix A is 4-dimensional, what is the dimension of the column space of A?*

We know $\operatorname{rank} A + \dim \operatorname{Nul} A = 6$. Since $\dim \operatorname{Nul} A = 4$, that tells us that $\operatorname{rank} A = 2$. That's the dimension of the column space.

4.6.22 *Is it possible that all solutions of a homogeneous sytem of ten linear equations in twelve variables are multiples of one fixed nonzero solution? Discuss.*

A homogenous 10 equations in 12 variables corresponds to solving $A\mathbf{x} = \mathbf{0}$ for a 10×12 matrix (each equation gives a row, and each variable a column). The dimension of the row space is at most 10, since there are only ten rows. That means the rank is at most 10. Since $\operatorname{rank} A + \dim \operatorname{Nul} A = 12$, it must be that $\dim \operatorname{Nul} A$ is at least 2. But that means that the solutions aren't all multiples of one fixed nonzero solution (that would correspond to the nullspace being one-dimensional, after all).

Problems for W 10/21:

4.7.1 *Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$ be basis for a vector space V, and suppose that $\mathbf{b}_1 = 6\mathbf{c}_1 - 2\mathbf{c}_2$ and $\mathbf{b}_2 = 9\mathbf{c}_1 - 4\mathbf{c}_2$.*

(a) *Find the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$ (i.e. the matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{\mathcal{P}}$).* The formula (from class on Friday) says that

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{\mathcal{P}} = \left[[\mathbf{b}_1]_{\mathcal{C}} \; [\mathbf{b}_2]_{\mathcal{C}} \right].$$

Both of the columns of this are given to us in the problem, so we get

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{\mathcal{P}} = \begin{bmatrix} 6 & 9 \\ -2 & -4 \end{bmatrix}.$$

(b) *Find $[\mathbf{x}]_{\mathcal{C}}$ for $\mathbf{x} = -3\mathbf{b}_1 + 2\mathbf{b}_2$. (Use part (a).)*
They are telling us that

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}.$$

We know that

$$[\mathbf{x}]_{\mathcal{C}} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{\mathcal{P}} [\mathbf{x}]_{\mathcal{B}}.$$

Putting it together, we get

$$[\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix} 6 & 9 \\ -2 & -4 \end{bmatrix} \begin{bmatrix} -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}.$$

4.7.7 *Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$ be the bases for $\mathbb{R}^2$ listed below.*

$$\mathbf{b}_1 = \begin{bmatrix} 7 \\ 5 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} -3 \\ -1 \end{bmatrix}, \quad \mathbf{c}_1 = \begin{bmatrix} 1 \\ -5 \end{bmatrix}, \quad \mathbf{c}_1 = \begin{bmatrix} -2 \\ 2 \end{bmatrix}.$$

Find the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$ and from $\mathcal{C}$ to $\mathcal{B}$.

We have a formula for just this situation (switching between two bases for $\mathbb{R}^n$):

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{\mathcal{P}} = \mathcal{P}_{\mathcal{C}}^{-1}\mathcal{P}_{\mathcal{B}}.$$

Plugging in the numbers, we get

$$\begin{aligned}
\mathcal{P}_{\mathcal{B}} &= \begin{bmatrix} 7 & -3 \\ 5 & -1 \end{bmatrix} \\
\mathcal{P}_{\mathcal{C}} &= \begin{bmatrix} 1 & -2 \\ -5 & 2 \end{bmatrix} \\
\mathcal{P}_{\mathcal{C}}^{-1} &= \frac{1}{(1)(2) - (-2)(-5)} \begin{bmatrix} 2 & 2 \\ 5 & 1 \end{bmatrix} = -\frac{1}{8} \begin{bmatrix} 2 & 2 \\ 5 & 1 \end{bmatrix} \\
\underset{\mathcal{C}\leftarrow\mathcal{B}}{\mathcal{P}} &= \mathcal{P}_{\mathcal{C}}^{-1}\mathcal{P}_{\mathcal{B}} = -\frac{1}{8} \begin{bmatrix} 2 & 2 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} 7 & -3 \\ 5 & -1 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ -5 & 2 \end{bmatrix}
\end{aligned}$$

We also want to know $\underset{\mathcal{B}\leftarrow\mathcal{C}}{\mathcal{P}}$. Luckily for us,

$$\underset{\mathcal{B}\leftarrow\mathcal{C}}{\mathcal{P}} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{\mathcal{P}}^{-1} = \begin{bmatrix} -3 & 1 \\ -5 & 2 \end{bmatrix}^{-1} = \frac{1}{(-6) - (-5)} \begin{bmatrix} 2 & -1 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -5 & 3 \end{bmatrix}.$$

You could also find this by computing $\mathcal{P}_{\mathcal{B}}^{-1}\mathcal{P}_{\mathcal{C}}$; reassuringly, you end up with the same answer either way.

4.7.9 *Again, with*

$$\mathbf{b}_1 = \begin{bmatrix} -6 \\ -1 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \quad \mathbf{c}_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \quad \mathbf{c}_1 = \begin{bmatrix} 6 \\ -2 \end{bmatrix}.$$

Not a lot different from the one above:

$$\begin{aligned}
\mathcal{P}_{\mathcal{B}} &= \begin{bmatrix} -6 & 2 \\ -1 & 0 \end{bmatrix} \\
\mathcal{P}_{\mathcal{C}} &= \begin{bmatrix} 2 & 6 \\ -1 & -2 \end{bmatrix} \\
\mathcal{P}_{\mathcal{C}}^{-1} &= \frac{1}{(2)(-2) - (6)(-1)} \begin{bmatrix} -2 & -6 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} -1 & -3 \\ 1/2 & 1 \end{bmatrix} \\
\underset{\mathcal{C}\leftarrow\mathcal{B}}{\mathcal{P}} &= \mathcal{P}_{\mathcal{C}}^{-1}\mathcal{P}_{\mathcal{B}} = \begin{bmatrix} -1 & -3 \\ 1/2 & 1 \end{bmatrix} \begin{bmatrix} -6 & 2 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 9 & -2 \\ -4 & 1 \end{bmatrix}
\end{aligned}$$

4.7.13 *In $\mathbb{P}_2$, find the change-of-coordinates matrix from the basis*

$$\mathcal{B} = \left\{1 - 2t + t^2, 3 - 5t + 4t^2, 2t + 3t^2\right\}$$

to the standard basis $\mathcal{C} = \{1, t, t^2\}$. Then find the $\mathcal{B}$-coordinate vector for $-1 + 2t$.

In the coordinates, we have

$$\mathcal{P}_{\mathcal{B}} = \begin{bmatrix} 1 & 3 & 0 \\ -2 & -5 & 2 \\ 1 & 4 & 3 \end{bmatrix}$$

This gives

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{\mathcal{P}} = \mathcal{P}_{\mathcal{B}} = \begin{bmatrix} 1 & 3 & 0 \\ -2 & -5 & 2 \\ 1 & 4 & 3 \end{bmatrix}$$

The inverse is

$$\underset{\mathcal{B}\leftarrow\mathcal{C}}{\mathcal{P}} = \mathcal{P}_{\mathcal{B}} = \begin{bmatrix} -23 & -9 & 6 \\ 8 & 3 & -2 \\ -3 & -1 & 1 \end{bmatrix}$$

We want

$$[\mathbf{p}]_{\mathcal{B}} = \begin{bmatrix} -23 & -9 & 6 \\ 8 & 3 & -2 \\ -3 & -1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}.$$

Problems for F 10/23:

4.9.1 *A small remote village receives radio broadcasts from two radio stationts, a news station and a music station. Of the listeners who are tuned to the news station, 70% will remain listening to the news after the break that occurs each half hour, while 30% will switch to the music station at the station break. Of the listeners who are tuned to the music station, 60% will switch at the break, while 40% will remain. Suppose that at 8:15 AM, everyone is listening to the news.*

(a) *Give the stochastic matrix that describes how the radio listeners tend to change stations at ecah station break. Label the rows and columns.*

The matrix is:

$$A = \begin{bmatrix} 0.7 & 0.6 \\ 0.3 & 0.4 \end{bmatrix}$$

(b) *Give the initial state vector.*

Everyone is listening to the news, so it's

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

(c) *What percentage of the listeners will be listening to the music station at 9:25 AM, after breaks at 8:30 and 9:00 AM?*

To find how things change after each break, we need to apply the matrix A. Since there are two breaks, our answer is:

$$\begin{bmatrix} 0.7 & 0.6 \\ 0.3 & 0.4 \end{bmatrix}^2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.67 \\ 0.33 \end{bmatrix}$$

4.9.4 *The weather in Columbus is either good, indifferent, or bad on any given day. If the weather is good today, there is a 60% chance that the weather will be good tomorrow, 30% that it's indifferent, and 10% that it's bad. If it's indifferent today, it's good tomorrow with probability 40%, indifferent 30%. If bad today, it's good with probability 40% and indifferent 50%.*

(a) *What is the stochastic matrix for this situation?*

The stochastic matrix is

$$A = \begin{bmatrix} 0.6 & 0.4 & 0.4 \\ 0.3 & 0.3 & 0.5 \\ 0.1 & 0.3 & 0.1 \end{bmatrix}.$$

(b) *Suppose there is a 50% chance of good weather today and 50% of indifferent. What are the chances of bad weather tomorrow?*

Our new probability vector is

$$\begin{bmatrix} 0.6 & 0.4 & 0.4 \\ 0.3 & 0.3 & 0.5 \\ 0.1 & 0.3 & 0.1 \end{bmatrix} \begin{bmatrix} 0.5 \\ 0.5 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.3 \\ 0.2 \end{bmatrix}.$$

The chances of bad weather are the last entry: 20%.

(c) *Suppose the predicted weather for Monday is 40% indifferent and 60% bad. What are the chances for good weather on Wednesday?*

This is

$$\begin{bmatrix} 0.6 & 0.4 & 0.4 \\ 0.3 & 0.3 & 0.5 \\ 0.1 & 0.3 & 0.1 \end{bmatrix} \begin{bmatrix} 0 \\ 0.4 \\ 0.6 \end{bmatrix} = \begin{bmatrix} 0.480 \\ 0.336 \\ 0.184 \end{bmatrix}$$

So the chance of good weather is 48%.

4.9.7 *Find the steady-state vector for the following stochastic matrix:*

$$A = \begin{bmatrix} 0.7 & 0.1 & 0.1 \\ 0.2 & 0.8 & 0.2 \\ 0.1 & 0.1 & 0.7 \end{bmatrix}.$$

The steady state is a solution of $A\mathbf{x} = \mathbf{x}$, so we want $(A - I)\mathbf{x} = \mathbf{0}$. The matrix $A - I$ is just

$$A = \begin{bmatrix} -0.3 & 0.1 & 0.1 \\ 0.2 & -0.2 & 0.2 \\ 0.1 & 0.1 & -0.3 \end{bmatrix}$$

Row reduction gives rref as

$$R = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

The last variable is free, setting it to 1 we obtain a solution

$$\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

We want the steady state to be a probability vector, so we have to divide everything through by 4.

$$\begin{bmatrix} 1/4 \\ 1/2 \\ 1/4 \end{bmatrix}.$$

4.9.11 *Find the steady state for the Markov chain in exercise 4.9.1 (the radio thing). At some time late in the day, what fraction of the listeners will be listening to the news?*

Remember that

$$A = \begin{bmatrix} 0.7 & 0.6 \\ 0.3 & 0.4 \end{bmatrix}$$

so

$$A - I = \begin{bmatrix} -0.3 & 0.6 \\ 0.3 & -0.6 \end{bmatrix}$$

Solving $(A - I)\mathbf{x} = \mathbf{0}$, we obtain

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

This should be a probability vector, so divide through by 3:

$$\begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix}.$$

This means that late in the day, about 2/3 of people will be listening to the news. This was already almost the case after just two breaks!