1. Let x and y be two closed points in $\mathbb{P}^2_{\mathbb{C}}$.

   (a) Let $d \geq 1$ and $m_1, m_2 \geq 0$ be integers. Describe the coherent sheaf $\mathcal{F}_{d,m_1,m_2} = \mathcal{O}_{\mathbb{P}^2}(d) \otimes \mathcal{I}_x^{\otimes m_1} \otimes \mathcal{I}_y^{\otimes m_2}$. What are the global sections?

   Note that $\mathscr{I}_V = \mathscr{I}_x^{\otimes} m_1 \otimes \mathscr{I}_y^{\otimes} m_2$ is an ideal sheaf on X, and so determines a closed subscheme $i : V \to \mathbb{P}^2$. As a topological space V is just the two points x and y, but it comes with some extra thickening.

   Now, there's an exact sequence

   $$0 \longrightarrow \mathscr{I}_V \longrightarrow \mathcal{O}_{\mathbb{P}^2} \longrightarrow i_* \mathcal{O}_V \longrightarrow 0$$

   Twisting by an invertible sheaf preserves exactness of sequences of sheaves: locally, this is just twisting by $\mathcal{O}_X|_U$, which is an isomorphism. And exactness of sequences of sheaves is a local property. So we also have an exact sequence

   $$0 \longrightarrow \mathscr{I}_V(d) \longrightarrow \mathcal{O}_{\mathbb{P}^2}(d) \longrightarrow i_* \mathcal{O}_V(d) \longrightarrow 0$$

   The global sections functor is left-exact, so we get

   $$0 \longrightarrow \Gamma(\mathbb{P}^2, \mathscr{I}_V(d)) \longrightarrow \Gamma(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d)) \longrightarrow \Gamma(\mathbb{P}^2, i_* \mathcal{O}_V(d))$$

   (this isn't exact on the right in general, even in this setting!)

   Since $\mathscr{I}_V(d)$ is a subsheaf of $\mathcal{O}_{\mathbb{P}^2}(d)$, we can think of $\Gamma(\mathbb{P}^2, \mathscr{I}_V(d))$ as a subspace of $\Gamma(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d))$. Explicitly, the sequence identifies $\Gamma(\mathbb{P}^2, \mathscr{I}_V(d))$ as the kernel of the map $\Gamma(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d)) \to \Gamma(\mathbb{P}^2, i_* \mathcal{O}_V(d))$. This maps sends a degree d polynomial to its class in $\mathcal{O}_{\mathbb{P}^2,x}/\mathfrak{m}_x^{m_1} \oplus \mathcal{O}_{\mathbb{P}^2,y}/\mathfrak{m}_x^{m_2}$. For example, if $m_1 = m_2 = 2$, the map sends a polynomial to the data of its constant and linear terms at x and y. (You worked out a similar example on the first homework.)

   The conclusion is that $\Gamma(\mathbb{P}^2, \mathscr{I}_V(d))$ can be identified with the space of degree d homogeneous polynomials on $\mathbb{P}^2$ that vanish to order m_1 at x and order m_2 at y. The dimension of this space can be computed exactly, but I won't do it.

   (b) Let $d = 1$ and $m_1 = m_2 = 1$. Compute $\Gamma(\mathbb{P}^2, \mathcal{F}_{1,1,1})$. Is this sheaf globally generated?

   Now that we have the description above, we can figure this out. $\Gamma(\mathbb{P}^2, \mathcal{F}_{1,1,1})$ is the set of degree 1 homogeneous polynomials that give 0 when you restrict them to the closed subschemes x and y. There is a unique line between two points, so $\Gamma(\mathbb{P}^2, \mathcal{F}_{1,1,1})$ has dimension 1: a generating section is given by the defining equation for the line.

   This isn't globally generated: if z is another point on the line between x and y, then the unique section in $\Gamma(\mathbb{P}^2, \mathcal{F}_{1,1,1})$ vanishes in the stalk at z. Hence this stalk is not generated by global sections.

(c) The sheaf $\mathcal{F}_{1,0,0} = \mathcal{O}(1)$ is very ample on $\mathbb{P}^2$, so we know that $\mathcal{F}_{1,1,1} \otimes \mathcal{O}(1)^{\otimes k}$ is globally generated for some k. What k is needed?

I claim that $k = 1$ is enough. Then we're looking at $\mathcal{F}_{2,1,1}$, whose global sections correspond to quadrics vanishing at the two points x and y. This has $\Gamma(\mathbb{P}^2, \mathcal{F}_{2,1,1}) = 4$, and given any point z distinct from x and y, it's easy to find one that doesn't vanish at z. For example, take your section to have zero set the line between x and y, together with some other line not through z.

We also need to show that the stalks at x and y are generated by global sections. This is a little tricker: $\mathcal{F}_{2,1,1}$ isn't locally free, and these stalks are rank-2 modules over $\mathcal{O}_{X,x}$.

For concreteness, let's look at a standard affine chart around x, with coordinates X_0 and X_1. The stalk is generated by X_0 and X_1 as an $k[X_0, X_1]_{(X_0,X_1)}$-module (look at the local ring at the point).

To get global generation, we need to find a section which gives X_0 in this stalk, and a section which gives X_1. This means we want a quadric whose defining equation, when restricted to x, has leading term X_0, and this is clearly possible: again we can take a line through x with leading term X_0, together with any other line. Likewise we can get a quadric with leading term X_1. The same thing can then be done at the point y (remember that we don't care if the same sections generate the stalks at both x and y, so there's nothing to worry about).