Math 553, Lesieutre
Problem set #9
due March 18, 2016

1. Check that if A is a B-algebra and $I = \ker(A \otimes_B A \to A)$, then $A \to I/I^2$ given by $a \mapsto a \otimes 1 - 1 \otimes a$ is a derivation.

2. II.8.3

3. II.8.4

4. II.8.5

5. Let X be a nonsingular projective variety over k. The nth plurigenus $P_n(X)$ is equal to $\dim H^0(X, \omega_X^{\otimes n})$. Show that P_n is a birational invariant for any $n > 0$.

   Suppose that C is a curve of genus g. Compute $P_n(C)$ for $n \geq 0$. (You will need to use some facts about curves that may not be familiar – ask me if you need some hints. You can use the Riemann–Roch theorem, and the fact that $\deg \omega_C = 2g - 2$.

6. (Optional, since I added it late, but it's good to do a local calculation)

   We know how to compute the canonical class of a hypersurface in $\mathbb{P}^n$ using the adjunction formula. This can also be done by writing down a form in local coordinates.

   Suppose that $X \subset \mathbb{P}^{n+1}$ is a smooth hypersurface defined by the equation $F(x_0, \dots, x_{n+1}) = 0$. Let's consider the open set U_0 where $x_0 \neq 0$. The defining equation for X in U_0 is of the form $G(x_{10}, \dots, x_{n+1,0}) = 0$, where $x_{i0} = x_i/x_0$.

   Let $V_i \subset U_0$ be the open set $U_0 \cap U_i$, i.e. the points for which the coordinate x_{i0} is nonzero.

   A basis for $\Gamma(V_i, \omega_X|_{V_i})$ is given by the single form

$$\frac{(-1)^i}{\partial G/\partial y_i} dy_1 \wedge \cdots \wedge \widehat{dy_i} \wedge \cdots dy_{n+1}.$$

   (a) Check that $\omega_i = \omega_j$ on the overlap $V_i \cap V_j$. Hence the ω_i piece together to a global n-form on the set $U \subset X$

   (b) The form ω is regular on U, but it has other poles outside of this set. Compute the divisor of ω. What is the canonical class of X?

   (c) Compute the geometric genus $\dim \Gamma(X, \omega_X)$.

   (d) Use your answer to conclude that two hypersurfaces X, X' in $\mathbb{P}^{n+1}$ of degrees at least $n + 1$ are not birational.