Utah Summer School on Higher Dimensional Algebraic Geometry
Problem session #1: Dynamical degrees & entropy
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July 18, 2016

Problem 1. a) Let $X = \mathbb{P}^1$ and let $\phi : X \to X$ be the map $z \mapsto z^d$. Compute the entropy $h_{\mathrm{top}}(\phi)$ in two different ways. Do your answers agree?

i) Directly from the topological definition;

ii) Using the Gromov-Yomdin theorem.

b) Show that no orbit of ϕ is dense for the usual topology, but that there exist points whose orbit is dense in S^1, and therefore Zariski-dense in $\mathbb{P}^1$.

c) Let E be an elliptic curve, and let M be an element of $\mathrm{SL}_2(\mathbb{Z})$ with $\phi_M : E \times E \to E \times E$ the induced automorphism. Repeat part (a) for this map. (You can try the example $M = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$).

Problem 2. Show that $\mathbb{P}^1 \times \mathbb{P}^1$ does not admit an automorphism of positive entropy. Can you prove this for any other classes of varieties?

In dimension 2, the dynamical degrees detect the existence of invariant fibrations for a birational map, according to the following theorem of Diller and Favre.

Theorem 1. Let X be a projective surface and let $f : X \to X$ be an automorphism. Then one of the following holds:

1. f^* is of finite order; in this case some iterate of f is isotopic to the identity (i.e. some $f^{\circ k}$ lies in $\mathrm{Aut}^0(X)$),
2. some iterate of f^* is unipotent of infinite order; in this case $\lambda_1(f) = 1$ and f preserves a fibration $\pi : X \to C$ onto a curve.
3. f^* is semi-simple; in this case $\lambda_1(f)$ is a Salem number (i.e. an algebraic integer whose conjugates over $\mathbb{Q}$ are $1/\lambda_1(f)$ and some complex numbers of modulus 1) and f does not preserve any fibration.

We won't give a full proof, but the next few problems demonstrate some aspects of this fact.

Problem 3. a) Suppose that $f : X \to X$ is an automorphism of a smooth projective surface, and there is a fibration to a curve $\pi : X \to C$ and an automorphism $g : C \to C$ with $\pi \circ f = g \circ \pi$. Show that $\lambda_1(f) = 0$.

b) Let X be the blow-up of $\mathbb{P}^2$ at the base locus of a pencil of cubics. Show that every automorphism of X must preserve the resulting elliptic fibration, and so has entropy 0.

Problem 4. Let X, f be as in Theorem 1 and let

$$\mathcal{C} = \{D \in N^1_{\mathbb{R}}(X), D.D \geq 0\}$$

be the positive cone for the intersection product.

a) Show that f^* preserves a line in $\mathcal{C}$.

b) Show that, if f^* preserves a line in the interior of $\mathcal{C}$, then f^* has finite order.

c) Show that if f^* preserves a single line in $\partial\mathcal{C}$, then some iterate of f^* is unipotent of infinite order, and that then $||(f^n)^*||$ grows as cn^2.

d) Show that if f^* preserves at least two lines in $\partial\mathcal{C}$ and no line in the interior of $\mathcal{C}$, then f^* is semi-simple and $||(f^n)^*||$ grows as $c\lambda^n$, where $\lambda = \lambda_1(f)$ is a Salem number. Furthermore, f^* preserves exactly two lines in $\mathcal{C}$, which are not defined over $\mathbb{Q}$.

Problem 5. Let's show that if $\phi : X \to X$ is an automorphism with unbounded degree, then it preserves an elliptic fibration. You might want to assume X is a K3 surface the first time through.

a) Show that there exists an integral nef class D with $\phi^* D = D$, $D^2 = 0$, $D \cdot K_X = 0$.

b) Show that $h^0(nD) \geq 2$ for sufficiently large n.

c) Show that the rational map determined by nD is an elliptic fibration.

Problem 6. Let E be an elliptic curve, and let M be an element of $\mathrm{SL}_2(\mathbb{Z})$ with $\phi_M : E \times E \to E \times E$ the induced automorphism. Describe the induced linear automorphism f^* and check the results of Theorem 1. In the semi-simple case, show that ϕ_M preserves a pair of smooth foliations $\mathcal{F}_+, \mathcal{F}_-$ whose leaves are dense in $E \times E$.

Problem 7. Fix a smooth plane cubic $E \subset \mathbb{P}^2$, and let p be a general point on E. We may define a rational map $\tau_p : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ as follows. Given a point $x \in \mathbb{P}^2$, draw the line L from x to p. Generically, this line meets E in two points. There is a unique involution of L that fixes these two points (why?). Let $\tau_p(x)$ be the image of x under this involution.

a) The map τ_p is a rational map, not an automorphism. Determine the indeterminacy locus of τ_p. Show that after blowing up the indeterminacy locus, we obtain a map $\tilde{\tau}_p : X_p \to X_p$ which is an involutive automorphism of a rational surface (or just trust us and go to (b)).

b) Compute the action of $\tilde{\tau}_p : X_p \to X_p$ on $N^1(X_p)$.

c) Show that if $q \in E$ is another point, there exists a rational surface X_{pq} such that τ_p and τ_q both lift to automorphisms of X_{pq}. Compute the matrices for the action of these involutions on $N^1(X_{pq})$.

d) Check that $\tau_p \circ \tau_q$ acts with infinite order on $N^1(X_{pq})$, but this map has entropy 0.

e) This means that $\tau_p \circ \tau_q$ preserves an elliptic fibration: can you find it?

f) Fix a third point r on E. Using the same construction, exhibit a positive entropy automorphism of a rational surface X_{pqr}.

Problem 8. a) Construct a compact metric space X and a map $\phi : X \to X$ with $h_{\text{top}}(\phi) = \infty$. Can you find such a map when $X = [0, 1]$?

b) Show (from the definition) that if $\phi : X \to X$ is an automorphism of a variety, $h_{\text{top}}(\phi)$ is finite. (Hint: in fact, show that if $\phi : X \to X$ is Lipschitz with constant C on a manifold X, then $h_{\text{top}}(\phi) \leq C \dim X$.)