Utah Summer School on Higher Dimensional Algebraic Geometry
Problem session #1: Dynamical degrees & entropy
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Problem 1. a) *Let $X = \mathbb{P}^1$ and let $\phi : X \to X$ be the map $z \mapsto z^d$. Compute the entropy $h_{top}(\phi)$ in two different ways. Do your answers agree?*

i) *Directly from the topological definition;*

We first show that $h_{\text{top}}(\phi) \geq \log d$: the unit circle is preserved by ϕ, therefore $h_{\text{top}}(\phi) \geq h_{\text{top}}(\phi|_{S^1})$. Now, identifying S^1 with $\mathbb{R}/\mathbb{Z}$, the action of $\phi|_{S^1}$ is $[\alpha] \mapsto [d \cdot \alpha]$, where $[\alpha]$ is the class of α modulo $\mathbb{Z}$. Writing the real number α in base d

$$\alpha = c_1 d^{-1} + c_2 d^{-2} + \dots$$

this corresponds to erasing the first digit and rescaling:

$$[d \cdot \alpha] = [c_2 d^{-1} + c_3 d^{-2} + \dots].$$

Now it is easy to show that, for $\epsilon = d^{-M}$, the number of (N, ϵ)-separated orbits is exactly d^{M+N}, and that therefore

$$h_{\text{top}}(\phi) \geq h_{\text{top}}(\phi|_{S^1}) = \log d.$$

To show the converse inequality, remark first that all orbits starting outside the unit circle converge either to 0 or to ∞. One can bound the number of (N, ϵ)-separated orbits in the interior of the unit ball by noticing that, for $r, r' < 1$,

$$|re^{i\theta} - r'e^{i\theta'}| \leq C(|r - r'| + d(\theta, \theta'))$$

for some constant $C > 0$ (here d is the standard distance on $\mathbb{R}/2\pi\mathbb{Z}$). This means that, since for $|z| < 1$ we have $|\phi^N(z)| \to 0$ as $N \to +\infty$, for big N the number of (N, ϵ)-separated orbits for ϕ and for $\phi|_{S^1}$ are essentially the same. The same argument can be applied to bound the number of orbits starting outside the unit ball.

ii) *Using the Gromov-Yomdin theorem.*

Since $H^0(\mathbb{P}^1, \mathbb{C})$ is the additive group of constant complex-valued functions on $\mathbb{P}^1$, the action of ϕ on it is trivial. Therefore in order to apply Gromov-Yomdin result we just need to compute the action of ϕ on $H^2(X, \mathbb{C})$ (or equivalently on $Pic(\mathbb{P}^1)$). Since $H^2(X, \mathbb{C})$ is generated by the class of a point $[p]$ and $\phi^*([p]) = d[p]$, the action of ϕ is the multiplication by d; the result follows from Gromov-Yomdin theorem.

b) *Show that no orbit of ϕ is dense for the usual topology, but that there exist points whose orbit is dense in S^1, and therefore Zariski-dense in $\mathbb{P}^1$.*

All the orbits either are contained in S^1 or converge to 0 or ∞. We can eplicitly construct a point α of $S^1 \cong \mathbb{R}/\mathbb{Z}$ whose orbit is dense in S^1: in base d we write

$$\alpha = 0.c_1 c_2 c_3 \dots$$

in such a way that every finite sequence of digits appears in the sequence (c_i). Since ϕ acts on $\mathbb{R}/\mathbb{Z}$ by erasing the first digit and then rescaling, this condition ensures that the orbit of α is dense in S^1.

c) *Let E be an elliptic curve, and let M be an element of* $\mathrm{SL}_2(\mathbb{Z})$ *with* $\phi_M : E \times E \to E \times E$ *the induced automorphism. Repeat part (a) for this map. (You can try the example* $M = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$*).*

Let x, y be complex coordinates on the two copies of E in X. The matrix

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

acts on $X = E \times E$ as $\phi_M(x, y) = (ax + by, cx + dy)$.

1. Suppose for simplicity that M is diagonalizable and pick a basis of eigenvectors (v_1, v_2). Remark that the eigenvalues have to be real, so that we can suppose v_1 and v_2 to be real as well. By cutting the torus X into small cubes in the directions v_1, iv_1, v_2, iv_2 one can find precise bounds on the number of (N, ϵ)-separated points and obtian the same result as in Gromov-Yomdin (see next point).
2. The cohomology space $H^{1,0}(X)$ is generated by dx, dy, and the action of ϕ_M^* on this basis is given by the matrix M^T. Let λ, λ^{-1} be the eigenvalues of M; since $H^{1,1}(X) = H^{1,0}(X) \otimes H^{0,1}(X)$, the eigenvalues of $\phi_M^* \colon H^{1,1}(X) \to H^{1,1}(X)$ are $|\lambda|^2$, $|\lambda|^{-2}$ and 1 (with multiplicity 2). Therefore, by Gromov-Yomdin,

$$h_{\mathrm{top}}(\phi_M) = 2 \log |\lambda|,$$

   where λ is the maximal modulus eigenvalue of M.

Problem 2. *Show that* $\mathbb{P}^1 \times \mathbb{P}^1$ *does not admit an automorphism of positive entropy. Can you prove this for any other classes of varieties?*

There are a couple ways you could approach this. For one, we know what the automorphism group of $\mathbb{P}^1 \times \mathbb{P}^1$ is, and it's possible to just check that all these maps have entropy equal to 0.

A slicker approach is this: think directly about the action of automorphisms on $N^1(X)$, which is a 2-dimensional vector space. Any automorphism pulls back effective classes to effective classes, and so preserves the effective cone $\overline{\mathrm{Eff}}(X)$. In particular, the generators of the two extremal rays on $\overline{\mathrm{Eff}}(X)$ are either both fixed, or both exchanged. Either way, the automorphism ϕ^2 fixes both generators of the effective cone, and so acts be the identity on $N^1(X)$. This means that it has entropy 0, according to the Gromov–Yomdin theorem.

The same argument works on any variety for which the effective cone is a rational polyhedral cone (i.e. generated by some finite set of generators). This holds for many varieties, for example any Mori Dream Space (and hence any Fano variety).

Another easy category of varieties with no positive entropy automorphisms are varieties with Picard rank 1. In this case $N^1(X)$ is one-dimensional, and since an automorphism must fix the generator, the map has entropy 0.

In dimension 2, the dynamical degrees detect the existence of invariant fibrations for a birational map, according to the following theorem of Diller and Favre.

Theorem 1. Let X be a projective surface and let $f \colon X \to X$ be an automorphism. Then one of the following holds:

1. f^* is of finite order; in this case some iterate of f is isotopic to the identity (i.e. some $f^{\circ k}$ lies in $\mathrm{Aut}^0(X)$),
2. some iterate of f^* is unipotent of infinite order; in this case $\lambda_1(f) = 1$ and f preserves a fibration $\pi\colon X \to C$ onto a curve.
3. f^* is semi-simple; in this case $\lambda_1(f)$ is a Salem number (i.e. an algebraic integer whose conjugates over $\mathbb{Q}$ are $1/\lambda_1(f)$ and some complex numbers of modulus 1) and f does not preserve any fibration.

We won't give a full proof, but the next few problems demonstrate some aspects of this fact.

Problem 3. a) *Suppose that $f : X \to X$ is an automorphism of a smooth projective surface, and there is a fibration to a curve $\pi : X \to C$ and an automorphism $g : C \to C$ with $\pi \circ f = g \circ \pi$. Show that $\lambda_1(f) = 0$.*

Suppose by contradiction that $\lambda = \lambda_1(f) > 1$; since f is an automorphism, this means that there exists a non-trivial class $D \in N^1_{\mathbb{R}}(X)$ such that $f^*D = \lambda D$. Let $F \in N^1(X)$ be the class of a fibre of π, so that $f^*F = F$ and $F.F = 0$. Then

$$D.D = f^*D.f^*D = \lambda^2 D.D, \qquad D.F = f^*D.f^*F = \lambda D.F$$

and therefore $D.D = D.F = 0$.

By Hodge's index theorem, the intersection product on $N^1_{\mathbb{R}}(X)$ has signature $(1, \rho(X) - 1)$; in particular, the maximal dimension of a subspace $V \subset N^1_{\mathbb{R}}(X)$ on which the intersection product is identically 0 is 1. Therefore we have $F.F \neq 0$, which contradicts the definition of F.

b) *Let X be the blow-up of $\mathbb{P}^2$ at the base locus of a pencil of cubics. Show that every automorphism of X must preserve the resulting elliptic fibration, and so has entropy* 0.

Let $p_1, \ldots, p_n$ be the base points of the pencil, and let $E_1, \ldots, E_n$ be the corresponding exceptional divisors of the blow-up $\pi\colon X \to \mathbb{P}^2$. Any automorphism of X preserves the canonical divisor $K_X = \pi^*\mathcal{O}(-3) + \sum_{i=1}^n E_i$. The strict transform of a cubic of the pencil has linear class $-K_X$ in $\mathrm{Pic}(X)$; therefore, the image of the strict transform of a cubic in the pencil is again a global section of $-K_X$. Since two strict transforms are disjoint, we have $K_X^2 = 0$; in particular, any global section of $-K_X$ must be the strict transform of an element of the pencil (otherwise the self-intersection of $-K_X$ would be positive). This shows that any automorphism of X preserves the elliptic fibration whose fibres are the strict transforms of the cubics in the pencil.

Problem 4. *Let X, f be as in Theorem 1 and let*

$$\mathcal{C} = \{D \in N^1_{\mathbb{R}}(X), D.D \geq 0\}$$

be the positive cone for the intersection product.

a) *Show that f^* preserves a line in $\mathcal{C}$.*

By Hodge's index theorem, the intersection product on $N^1_{\mathbb{R}}(X)$ has signature $(1, \rho(X)-1)$, i.e. there exist a linear coordinates $x_1, \dots x_n$ such that the product is $q(x) = x_1^2 - (x_2^2 + \cdots + x_n^2)$. Therefore, the projectivization $\mathbb{P}\mathcal{C} \subset \mathbb{P}N^1_{\mathbb{R}}(X)$ is homeomorphic to an $(n-1)$-dimensional closed ball (to see this, one can cut with the affine hyperplane $\{x_1 = 1\}$).
The linear automorphism f^* preserves $\mathcal{C}$, and therefore induces a continuous homeomorphism of $\mathbb{P}\mathcal{C}$ onto itself. By Brouwer's fixed point theorem, any continuous map from the closed ball into itself admits a fixed point, which shows the claim.

b) *Show that, if f^* preserves a line in the interior of $\mathcal{C}$, then f^* has finite order.*

Let $\mathbb{R}D \subset \mathcal{C}$ be a fixed line in the interior of $\mathcal{C}$; then, since $D.D = f^*D.f^*D \neq 0$, we must have $f^*D = D$ ($f^*D = -D$ is impossible because the half cone of $\mathcal{C}$ containing the ample classes is f^*-invariant as well), and in particular we can suppose that D is an integer class. Therefore f^* preserves D and its orthogonal space $D^\perp$, which is a hyperplane defined over $\mathbb{Q}$ on which the intersection form is defined negative.
Now, any element of the orthogonal group preserving a lattice has finite order: to see this, take any base of the lattice $e_1, \dots e_{n-1}$; f^* must preserve the set of elements of the lattice with norm greater or equal than $\min\{e_1.e_1, \dots e_{n-1}.e_{n-1}\}$. This set being finite, some iterate of f^* acts as the identity on it, and therefore on the whole $D^\perp$. Since $f^*D = D$, such an iterate is the identity.

c) *Show that if f^* preserves a single line in $\partial\mathcal{C}$, then some iterate of f^* is unipotent of infinite order, and that then $||(f^n)^*||$ grows as cn^2.*

Before giving the solution we will prove a lemma that will be useful later.

Lemma 1. *Let $f\colon X \to X$ be an automorphism of a surface and let $f^*\colon N^1_{\mathbb{R}}(X) \to N^1_{\mathbb{R}}(X)$ be the induced linear automorphism. If λ is an eigenvalue of f^* with modulus $\neq 1$, then so is λ^{-1}; furthermore, λ and λ^{-1} both have algebraic multiplicity 1 and are the only eigenvalues with modulus $\neq 1$.*

The Lemma is essentially a corollary of the fact that isotropic subspaces of $N^1_{\mathbb{R}}(X)$ endowed with the intersection form have dimension at most 1. First remark that if λ is an eigenvalue with modulus $\neq 1$, then so is $\bar{\lambda}$; if we had $\lambda \notin \mathbb{R}$ and $v \in N^1_{\mathbb{C}}(X)$ is an eigenvector, then v and $\bar{v}$ are not collinear and by the above remark the values $v.v, v.\bar{v}, \bar{v}.\bar{v}$ cannot be simultaneously 0. But this leads to a contradiction with the formula $v.w = f^*v.f^*w$, so that λ has to be real.
Now let λ, μ be two (real) eigenvalues with modulus $\neq 1$ and let v, w be two eigenvectors. Again $v.v, v.w, w.w$ cannot be simultaneously 0, but $v.v = \lambda^2(v.v) = 0$ and $w.w = \mu^2(w.w) = 0$, so that $v.w = \lambda\mu(v.w) \neq 0$ and $\lambda\mu = 1$. A similar proof also shows that the multiplicity of λ and λ^{-1} is at most 1.

Now, suppose that there is no preserved line in the interior of $\mathcal{C}$ and exactly one preserved line on the boundary of $\mathcal{C}$. By the lemma, if at least one of the eigenvalues of f^* had modulus $\neq 1$, then we would have two such eigenvalues and they would be real, so we would have two or more preserved lines on the boundary of $\mathcal{C}$. Therefore all eigenvalues have modulus 1.
Since f^* preserves the lattice $N^1(X)$, the eigenvalues are algebraic integers of modulus 1 all of whose conjugates also have modulus 1. By Kronecker's lemma, they all have to be roots of unity, so that all the eigenvalues of some iterate of f^* are equal to 1 (which is the same as saying that such an iterate is unipotent).
Now, suppose by contradiction that the growth of $||(f^n)^*||$ is at least cn^3, i.e. the maximal Jordan block of f^* has dimension at least 4; this means that the restriction of f^* (or some

iterate) to some subspace of $N^1_{\mathbb{R}}(X)$ of dimension 4 is represented by the following matrix with respect to a basis v_1, v_2, v_3, v_4:

$$J = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$J^n \sim \begin{pmatrix} 1 & n & n^2/2 & n^3/6 \\ 0 & 1 & n & n^2/2 \\ 0 & 0 & 1 & n \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Now

$$v_4.v_4 = J^n v_4.J^n v_4 \sim \frac{1}{36}n^6(v_1.v_1) + \frac{1}{6}n^5(v_1.v_2) + \left(\frac{1}{4}(v_2.v_2) + \frac{1}{3}(v_1.v_3)\right) + \ldots,$$

so we must have

$$v_1.v_1 = v_1.v_2 = \frac{1}{4}(v_2.v_2) + \frac{1}{3}(v_1.v_3) = 0.$$

The same computation for $v_3.v_3$ leads to

$$(v_2.v_2) + \frac{1}{2}(v_1.v_3) = 0,$$

so that the restriction of the intersection form to the subspace generated by $v_, v_2$ is identically 0, a contradiction.

Now we only have to prove that the growth of $||(f^n)^*||$ cannot be linear. Now, f^* preserves the half cone $\mathcal{C}_+$ containing the ample classes. If by contradiction the growth is linear, pick any class v in the interior of the positive half-cone; there exists a class v_1 on the boundary of $\mathcal{C}$ such that $(f^n)^*v = v + nv_1$ for $n \in \mathbb{N}$, so that $v_1 \in \mathcal{C}_+$; but we also have $(f^{-n})^*v = v - nv_1$ for $n \in \mathbb{N}$, so that $v_1 \in \mathcal{C}_+$ as well, a contradiction.

NB: a similar proof shows that, in any dimension, if the growth of $||(f^n)^*||$ is polynomial, then the degree of polynomial growth is even.

d) *Show that if f^* preserves at least two lines in $\partial\mathcal{C}$ and no line in the interior of $\mathcal{C}$, then f^* is semi-simple and $||(f^n)^*||$ grows as $c\lambda^n$, where $\lambda = \lambda_1(f)$ is a Salem number. Furthermore, f^* preserves exactly two lines in $\mathcal{C}$, which are not defined over $\mathbb{Q}$.*

If two preserved lines had eigenvalue 1, then f^* would act as the identity on the plane generated by these lines, and in particular f^* would preserve a line in the interior of $\mathcal{C}$, a contradiction. Therefore, by the Lemma above, f^* has exactly two eigenvalues with modulus $\neq 1$ (which are real): λ and λ^{-1}. Since they are algebraic integers, they cannot be rational and therefore their eigenspaces are not defined over $\mathbb{Q}$. Finally, suppose there is a third preserved line $\mathbb{R}v$ on the boundary of $\mathcal{C}$ (with eigenvalue 1 by the Lemma) and let $\mathbb{R}v_+$ be the eigenspace for the eigenvector λ. By Hodge's index theorem we have $v.v_1 \neq 0$, and we have $v.v_+ = f^*v.f^*v_+ = \lambda(v.v_+)$, which leads to a contradiction.

Problem 5. *Let's show that if $\phi : X \to X$ is an automorphism with unbounded degree, then it preserves an elliptic fibration. You might want to assume X is a K3 surface the first time through.*

The reference here is again Diller–Favre [DF01].

a) *Show that there exists an integral nef class D with $\phi^* D = D$, $D^2 = 0$, $D \cdot K_X = 0$.*
Consider the limit of the sequence of divisors

$$D = \lim_{n\to\infty} \frac{\phi^{n*}(H)}{\|\phi^{n*}(H)\|},$$

where H is a fixed ample and we have chosen a norm $\|\cdot\|$ on $N^1(X)$. The limit exists since after normalizing this is a sequence of divisors in a compact subset of $N^1(X)$. The limit is surely nef, since each $\phi^{n*}(H)$ is nef. Since we are assuming that $|\lambda_1(\phi)| = 1$, it must be that $\phi^* D = D$. The question is, why is this an integral class, rather than just an $\mathbb{R}$-divisor?
To see this, think about the Jordan decomposition of ϕ^*. The set of divisors E (maybe not ample) for which the limit in question is D is an open set in $N^1(X)$, by simple linear algebra. So the vector is nothing more than the leading eigenvector of the largest Jordan block of the matrix for ϕ^*, which is certainly integral.
Note that

$$0 = \lim_{n\to\infty} \frac{1}{\|\phi^{n*}H\|}(H \cdot K_X) = \lim_{n\to\infty} \frac{\phi^{n*}(H)}{\|\phi^{n*}(H)\|} \cdot K_X = .D \cdot K_X.$$

b) *Show that $h^0(nD) \geq 2$ for sufficiently large n.*
In the case of a K3, this is immediate from Riemann-Roch:

$$h^0(D) + h^0(K_X - D) = \chi(\mathcal{O}_X) + h^1(D) = 2 + h^1(D).$$

But on a K3 it is impossible that $h^0(K_X - D) \geq 1$, since this is $h^0(-D)$ and D is a nonzero nef class. Hence we obtain $h^0(D) \geq 2$.
The case of a rational surface requires a harder look. I can't quite follow the argument of [DF01] in this case; you may find it helpful to consult [Gri16].

c) *Show that the rational map determined by nD is an elliptic fibration.*
Passing to a multiple, we might as well assume that $h^0(D) \geq 2$, and we get a map $\phi_D : X \dashrightarrow \mathbb{P}^1$. After removing the fixed component of $|D|$, we may assume the map ϕ_D is a morphism. Taking the Stein factorization, we get a map $\pi : X \to \mathbb{P}^1$ whose general fibers are irreducible. The genus of these fibers is 1 by adjunction, and so we have an elliptic fibration which is ϕ-invariant.

Problem 6. *Let E be an elliptic curve, and let M be an element of $\mathrm{SL}_2(\mathbb{Z})$ with $\phi_M : E \times E \to E \times E$ the induced automorphism. Describe the induced linear automorphism f^* and check the results of Theorem 1. In the semi-simple case, show that ϕ_M preserves a pair of smooth foliations $\mathcal{F}_+, \mathcal{F}_-$ whose leaves are dense in $E \times E$.*

If M has finite order, then we are in situation (a) of the Theorem.

If M (or some iterate) is unipotent, then we are in situation (b) of the Theorem. In some base of $\mathbb{C}^2$ we can write

$$M = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

and then ϕ_M preserves the elliptic fibration whose fibres are given by $\{y = constant\}$.

Finally, if M is semi-simple, we have seen that $\lambda_1(\phi_M)$ is a Salem number (actually quadratic in this case). The foliations $\mathcal{F}_+, \mathcal{F}_-$ are given by the constant directions of the eigenspaces of M. The action on leaves of $\mathcal{F}_+$ (resp. $\mathcal{F}_-$) is a homothety with factor λ (resp. λ^{-1}), which easily implies that $\mathcal{F}_+, \mathcal{F}_-$ are the only invariant foliations; since their leaves have irrational slope, they aren't the fibres of an invariant fibration, so that there are no invariant fibrations.

Problem 7. *Fix a smooth plane cubic $E \subset \mathbb{P}^2$, and let p be a general point on E. We may define a rational map $\tau_p : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ as follows. Given a point $x \in \mathbb{P}^2$, draw the line L from x to p. Generically, this line meets E in two points. There is a unique involution of L that fixes these two points (why?). Let $\tau_p(x)$ be the image of x under this involution.*

The reference for this example is [Bla13], which also considers a higher-dimensional analog of this map.

a) *The map τ_p is a rational map, not an automorphism. Determine the indeterminacy locus of τ_p. Show that after blowing up the indeterminacy locus, we obtain a map $\tilde{\tau}_p : X_p \to X_p$ which is an involutive automorphism of a rational surface (or just trust us and go to (b)).*

Let's try to reason out where the map is not defined; for a proof using a local calculation, check out [Bla13]. The one thing that can go wrong in our geometric description is that the line L might meet E at only a single point, rather than two distinct points. In this case, the involution τ_p isn't defined. There are four lines L_i through p which are tangent to E, namely the translates of $-p$ by the half-periods of E. Let $p_1, \ldots, p_4$ be the four points where this tangency occurs.

Consider what happens on a line "very close" to L (in the complex topology, say). You can see that the line L_i must be contracted to the point p_i. The other way, the point p_i is blown up to the line L_i. After blowing up these four points, together with the point p itself, we obtain a morphism. (This is not a complete argument! See [Bla13]. But it's enough geometry for us to do the rest of the problem.)

b) *Compute the action of $\tilde{\tau}_p : X_p \to X_p$ on $N^1(X_p)$.*

The space $N^1(X_p)$ is 6-dimensional, generated by the classes H, E_0^p, E_1^p, E_2^p, E_3^p, and E_4^p. Unfortunately, it's not so clear what the images of *any* of these basis vectors are. On the other hand, if we can find the images of six spanning classes in $N^1(X_p)$, that's enough to find the matrix.

There are a few easy ones. We've seen geometrically that $E_i^p \leftrightarrow H - E_0^p - E_i^p$ (for $1 \leq i \leq 4$). We also know that the elliptic curve E is itself fixed (as it must be – it represents the anticanonical class of $X_\mathbf{p}$!) That's five classes: we still need one more. But a general line through the point p is also preserved, as is clear from the construction. This gives the six requisite classes, and it's now a straightforward matter of linear algebra to see that the matrix for the map is given with respect to the above basis by the map

$$\tau_p^* = \begin{pmatrix} 3 & 2 & 1 & 1 & 1 & 1 \\ -2 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & 0 & 0 & 0 \\ -1 & -1 & 0 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 & -1 & 0 \\ -1 & -1 & 0 & 0 & 0 & -1 \end{pmatrix}$$

It is probably healthy to do some sanity checks at this point and make sure we haven't gone totally off the rails. I invite you to check that $(\tau_p^*)^2 = 1$ and $\tau_p^* K_X = K_X$, as expected.
This shows that the image of a general line is a cubic with a node at p, which isn't so clear from our original description of the map. We also see that the image of the exceptional divisor E_0^p is the quadric passing through the 5 points implicated in this mess, which is at least plausible since it's a (-1)-curve.

c) *Show that if $q \in E$ is another point, there exists a rational surface X_{pq} such that τ_p and τ_q both lift to automorphisms of X_{pq}. Compute the matrices for the action of these involutions on $N^1(X_{pq})$.*
The map τ_p^* lifts to the blow-up, and it simply preserves each of the classes E_i^q. That means it's given by the matrix

$$\tau_p^* = \begin{pmatrix} 3 & 2 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -2 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The matrix for τ_q^* is likewise given by

$$\tau_q^* = \begin{pmatrix} 3 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & -1 \\ -1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 \end{pmatrix}$$

d) *Check that $\tau_p \circ \tau_q$ acts with infinite order on $N^1(X_{pq})$, but this map has entropy* 0.
The product matrix is

$$\tau_p^* \tau_q^* = \begin{pmatrix} 9 & 2 & 1 & 1 & 1 & 1 & 6 & 3 & 3 & 3 & 3 \\ -6 & -1 & -1 & -1 & -1 & -1 & -4 & -2 & -2 & -2 & -2 \\ -3 & -1 & -1 & 0 & 0 & 0 & -2 & -1 & -1 & -1 & -1 \\ -3 & -1 & 0 & -1 & 0 & 0 & -2 & -1 & -1 & -1 & -1 \\ -3 & -1 & 0 & 0 & -1 & 0 & -2 & -1 & -1 & -1 & -1 \\ -3 & -1 & 0 & 0 & 0 & -1 & -2 & -1 & -1 & -1 & -1 \\ -2 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & -1 \\ -1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 \end{pmatrix}$$

Does it have infinite order? With some computer assistance, we find that the Jordan decomposition is $\tau_p^* \tau_q^* = J \Lambda J^{-1}$, where

$$J = \begin{pmatrix} 4 & 6 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & -4 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -2 & 0 & -1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & -2 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -2 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & -2 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ -2 & -2 & 0 & -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 1 & -1 & -1 & 0 & 0 & 0 & 1 & 1 & 1 \\ -1 & -1 & 1 & -1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 \\ -1 & -1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & -1 & 0 \\ -1 & -1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\Lambda = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

The 3×3 Jordan block you see in the top left corner of Λ guarantees that the map has infinite order. However, there is no eigenvalue with norm larger than 1, so the entropy of the map is 0.

e) *This means that $\tau_p \circ \tau_q$ preserves an elliptic fibration: can you find it?*
The proof of the Diller–Favre/Gizatullin theorem above tells us what divisor must determine the elliptic fibration: it's the map given by the leading eigenvector of the 3×3 Jordan block. We

can read that off from the computation above, and conclude that the invariant elliptic fibration must be given by the linear system associated to

$$D = 4H - 2E_0^p - E_1^p - E_2^p - E_3^p - E_4^p - 2E_0^q - E_1^q - E_2^q - E_3^q - E_4^q.$$

This is the linear system of quartics in $\mathbb{P}^2$ with nodes and p and q, and passing through the other 8 blown up points.

It's easy to check that this numerical class is indeed preserved by $\tau_p^*\tau_q^*$. What's not so clear is that there is actually a pencil of such curves, or how to construct them (short of reading through the entire argument of Gizatullin). Here's one direct approach.

Let L denote the strict transform of the line joining p and q, and let E denote the elliptic curve. Observe that

$$\begin{aligned} &L \sim H - E_0^p - E_0^q \\ &E \sim 3H - E_0^p - E_1^p - E_2^p - E_3^p - E_4^p - E_0^q - E_1^q - E_2^q - E_3^q - E_4^q = -K_X \\ &F \sim L + E \end{aligned}$$

Now, there's a short exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}_X(L) \longrightarrow \mathcal{O}_X(L+E) \longrightarrow \mathcal{O}_E(L+E) \longrightarrow 0$$

In cohomology, this gives

$$0 \longrightarrow H^0(X, \mathcal{O}_X(L)) \longrightarrow H^0(X, \mathcal{O}_X(L+E)) \longrightarrow H^0(E, \mathcal{O}_E(L+E))$$

$$\longrightarrow H^1(X, \mathcal{O}_X(L)) \longrightarrow H^1(X, \mathcal{O}_X(L+E)) \longrightarrow H^1(E, \mathcal{O}_E(L+E)) \longrightarrow$$

The first term is 1-dimensional, since L is an irreducible curve of self-intersection -1 (we blew up 2 points on a line). The second is what we're trying to figure out: remember that $L+E \sim D$, and we're trying to show that $H^0(X, \mathcal{O}_X(D)) > 1$. The last term in the first row is a bit more work. We have $E \cdot E = -1$ (we blew up ten points on a cubic) and $E \cdot L = 1$. So $(L+E)|_E$ is a degree 0 line bundle on E, and it either has a section or doesn't, depending on whether or not it's 0 in $\mathrm{Pic}^0(E)$.

However, $(L+E)|_E$ is less mysterious than $L+E$ itself: we noticed earlier that there these is a quadric Q_p in $\mathbb{P}^2$ which is tangent to E at p, and passes through the points $p_1, \ldots, p_4$. This quadric restricts to E as a section of $(2H - 2E_0^p - E_1^p - E_2^p - E_3^p - E_4^p)|_E$. Note that it doesn't give a section in $H^0(X, \mathcal{O}_X(3H - 2E_0^p - E_1^p - E_2^p - E_3^p - E_4^p))$: the quadric doesn't have a double point at p, but it is tangent to the curve there. The sum with the corresponding quadric for q gives a section of $4H - 2E_0^p - E_1^p - E_2^p - E_3^p - E_4^p - 2E_0^q - E_1^q - E_2^q - E_3^q - E_4^q$, i.e. of D.

Hence the dimensions across the first row are

$$0 \longrightarrow 1 \longrightarrow ??? \longrightarrow 1 \longrightarrow$$

If we can show that $H^1(X, \mathcal{O}_X(L)) = 0$, we're in business: this would give $\dim H^0(X, \mathcal{O}_X(L+E)) = 2$, as required. To check it, write down the exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(L) \longrightarrow \mathcal{O}_L(L) \longrightarrow 0$$

In the long exact sequence, the group we want is sandwiched between $H^1(X, \mathcal{O}_X)$ (which is 0 since X is a rational surface), and $H^1(L, \mathcal{O}_L(L)) \cong H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1)) \cong H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1)) = 0$. Unwinding everything, we get $H^0(X, \mathcal{O}_X(D)) = 2$, as required. One section of the line bundle is the reduced curve given as the union of E and L, which is the image of the first map in our long exact sequence in cohomology. (I'm not sure how to see the second one geometrically.)

f) *Fix a third point r on E. Using the same construction, exhibit a positive entropy automorphism of a rational surface X_{pqr}.*

Now we get a blow-up of $\mathbb{P}^2$ at 15 points on E, and three involutions τ_p, τ_q, τ_r. I'm not going to write down all the matrices in this case, but the end result is that

$$\tau_p^*\tau_q^*\tau_r^* = \begin{pmatrix}
27 & 2 & 1 & 1 & 1 & 1 & 6 & 3 & 3 & 3 & 3 & 18 & 9 & 9 & 9 & 9 \\
-18 & -1 & -1 & -1 & -1 & -1 & -4 & -2 & -2 & -2 & -2 & -12 & -6 & -6 & -6 & -6 \\
-9 & -1 & -1 & 0 & 0 & 0 & -2 & -1 & -1 & -1 & -1 & -6 & -3 & -3 & -3 & -3 \\
-9 & -1 & 0 & -1 & 0 & 0 & -2 & -1 & -1 & -1 & -1 & -6 & -3 & -3 & -3 & -3 \\
-9 & -1 & 0 & 0 & -1 & 0 & -2 & -1 & -1 & -1 & -1 & -6 & -3 & -3 & -3 & -3 \\
-9 & -1 & 0 & 0 & 0 & -1 & -2 & -1 & -1 & -1 & -1 & -6 & -3 & -3 & -3 & -3 \\
-6 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & -1 & -4 & -2 & -2 & -2 & -2 \\
-3 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & -2 & -1 & -1 & -1 & -1 \\
-3 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & -2 & -1 & -1 & -1 & -1 \\
-3 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & -2 & -1 & -1 & -1 & -1 \\
-3 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & -2 & -1 & -1 & -1 & -1 \\
-2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & -1 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1
\end{pmatrix}$$

The characteristic polynomial is $(t-1)^4(t+1)^{10}(t^2 - 18t + 1)$, and the largest root is $t = 9 + 4\sqrt{5} \approx 17.94$. The entropy is then $\log(9 + 4\sqrt{5})$, which is positive. Phew.

Problem 8. a) *Construct a compact metric space X and a map $\phi : X \to X$ with $h_{top}(\phi) = \infty$. Can you find such a map when $X = [0,1]$?*

We know the map $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ given by $z \mapsto z^d$ has entropy $\log d$. Now let $X = \prod_{i=1}^{\infty} \mathbb{P}^1$, which is metrizable, and compact by Tychonoff's theorem. Consider the map $\psi : X \to X$ given by $z \mapsto z^i$ on the ith factor. One can now construct (n, ϵ)-separated sets of points by placing all the points in the ith factor, which gives a lower bound of $\log i$ on the entropy for any i. It follows that the entropy is infinite.

It is also possible to find such a map when $X = [0,1]$. One approach is to start with a sequence of functions $f_n : [0,1] \to [0,1]$ with $f_n(0) = 0$ and $f_n(1) = 1$ with $h_\top(f_n) \to \infty$. One can than use these to cook up a map of infinite entropy: define your function on $[0, 1/2]$ using f_1, then $[1/2, 3/4]$ using $f_2, \ldots$. There's a picture in [Mil, Figure 39], which is a nice source on the setup and basic properties of topological entropy.

b) *Show (from the definition) that if $\phi : X \to X$ is an automorphism of a variety, $h_{top}(\phi)$ is finite. (Hint: in fact, show that if $\phi : X \to X$ is Lipschitz with constant C on a manifold X,* *then* $h_{top}(\phi) \leq C \dim X$*.)*

You can find a proof of this one in Katok & Hasseblatt, page 120 [KH95].

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