Utah Summer School on Higher Dimensional Algebraic Geometry
Problem session #2: More dynamical degrees & examples
John Lesieutre and Federico Lo Bianco
July 19, 2016

Problem 1. a) *Prove that if $\phi : X \dashrightarrow X$ is a birational transformation and $n = \dim(X)$, then $\lambda_{n-d}(\phi) = \lambda_d(\phi^{-1})$.*
If ϕ is an automorphism, then

$$(\phi^N)^* H^{n-d}.H^d = H^{n-d}.(\phi^N)_* H^d = H^{n-d}.(\phi^{-N})^* H^d,$$

and we obtain the result by passing to the limit.
If ϕ is only birational, we can find dense open subsets U_N, V_N such that ϕ^N induces an isomorphism $U_N \cong V_N$; the same computations as above work provided we restrict to U_N, V_N, and give exactly the same result (for a general theory on the pull-back of cohomology classes by rational maps, see [Dem97]).

b) *Let $f\colon X \dashrightarrow X$ and $g\colon Y \dashrightarrow Y$ be (bi)rational maps and let $\pi\colon X \dashrightarrow Y$ be a generically finite map such that $\pi \circ f = g \circ \pi$. Show that $\lambda_p(f) = \lambda_p(g)$ for all p.*
Suppose first that π is an unramified covering of degree d and that f, g are biregular; if H_Y is an ample divisor, then so is $H_X := \pi^* H_X$, and we have

$$(f^N)^* H_X^p.H_X^{n-p} = \pi^*(g^N)^* H_Y^p.\pi^* H_Y^{n-p} = d \cdot \left((g^N)^* H_Y^p.H_Y^{n-p}\right),$$

so that the result follows by passing to the limit. In the general setting, one has to restrict to the dense open sets where the hypothesis we assumed stay true, and carry out the computations on those sets.

c) *Suppose that $\phi : X \to X$ is a positive entropy automorphism of a smooth threefold. Let D be a leading eigenvector of $\phi^* : N^1(X) \to N^1(X)$, and D' a leading eigenvector of $(\phi^{-1})^* : N^1(X) \to N^1(X)$. Show that either $D^2 = 0$ or $(D')^2 = 0$ (as elements of $N^2(X)$, or $H^{2,2}(X)$). (Hint: you can assume $\lambda_1(f)$ is a real eigenvalue)*
We have $\phi^* D = \lambda_1(f) D$ and $(f^{-1})^* D' = \lambda_1(f^{-1}) D' = \lambda_2(f) D'$ by point (a). If by contradiction we had $D^2 \neq 0 \neq (D')^2$, then since

$$f^* D^2 = \lambda_1(f)^2 D^2, \qquad (f^{-1})^*(D')^2 = \lambda_1(f^{-1})^2 (D')^2,$$

we would have $\lambda_2(f) \geq \lambda_1(f)^2$ and $\lambda_2(f^{-1}) = \lambda_1(f) \geq \lambda_1(f^{-1})^2 = \lambda_2(f)^2$, contradiction since $\lambda_1(f) > 1$.

Problem 2. *Suppose that $D \subset \mathbb{P}^3$ is a surface of degree d, with multiplicities m_1, m_2, m_3, and m_4 at the four coordinate points. Compute the degree and multiplicities of $\mathrm{Cr}(D)$, where Cr is the standard Cremona involution. What if D is a curve instead of a surface?*

Let me do this a couple different ways. The first is to attack it directly using the formula for the Cremona involution, which is $[W, X, Y, Z] \mapsto [W^{-1}, X^{-1}, Y^{-1}, Z^{-1}]$. Suppose our surface has degree d and is defined by an equation of the form

$$\sum_{i+j+k+l=d} a_{ijkl} W^i X^j Y^k Z^l$$

What's the multiplicity at each of the four standard coordinate points? To find the multiplicity at $[1, 0, 0, 0]$, you can pass to an affine chart where it's the origin by plugging in $W = 1$ to the formula. The multiplicity is then given by the lowest degree of any of the remaining terms in the sum. This term comes from the term in the original formula with the *largest* power of W, and we see that $m_1 = d - \mu_W$, where μ_W is the maximum power of W in any term. Similar formulas hold for the other multiplicities.

Now, the strict transform is defined by the formula

$$\sum_{i+j+k+l=d} a_{ijkl} \frac{1}{W^i X^j Y^k Z^l},$$

which doesn't really make sense until we clear denominators. To do this, we need to multiply through by $W^{\mu_W} X^{\mu_X} Y^{\mu_Y} Z^{\mu_Z}$. Observe that the resulting polynomial has degree

$$\begin{aligned} d' &= \mu_W + \mu_X + \mu_Y + \mu_Z - d \\ &= (d - m_1) + (d - m_2) + (d - m_3) + (d - m_4) - d = 3d - \sum_{i=1}^{4} m_i. \end{aligned}$$

What's the new multiplicity? Well, we can assume some term of the defining equation for our surface is not divisible by W. This means that the largest W factor in the equation for the strict transform is given by μ_W, and the new multiplicity is

$$m'_j = d' - \mu'_W = (3d - \sum_{i=1}^{4} m_i) - (d - m_j) = 2d - \sum_{i \neq j} m_i.$$

There is another, more geometric way to do the problem as well. What we're really trying to do is find the matrix for the map $N^1(X) \to N^1(X)$, where X is the blow-up of $\mathbb{P}^3$ at the four coordinate points. The map $X \dashrightarrow X$ is a pseudoautomorphism, and it suffices to find the images for a basis of $N^1(X)$.

There are four classes whose images we know: the exceptional divisor E_i is mapped to the strict transform of the plane through the points other than p_i, which has class $H - \sum_j E_j + E_i$. To get one more class, observe that the canonical class of X must be preserved by the pullback. This is enough information to find that the in the basis H, E_1, E_2, E_3, E_4, the matrix for the pullback is

$$\phi^* = \begin{pmatrix} 3 & 1 & 1 & 1 & 1 \\ -2 & 0 & -1 & -1 & -1 \\ -2 & -1 & 0 & -1 & -1 \\ -2 & -1 & -1 & 0 & -1 \\ -2 & -1 & -1 & -1 & 0 \end{pmatrix},$$

Reading across the rows, we recover the formulas we derived earlier.

Problem 3. a) *Compute the dynamical degrees for the following affine maps:* $(x, y) \mapsto (x^2y, xy)$, $(x, y) \mapsto (xy, y)$. *What about the maps* $(x, y) \mapsto (x^a y^b, x^c y^d)$ *more generally?*

The action of the space of matrices with integer coefficients on $\mathbb{P}^2$ by rational morphism given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(x, y) = (x^a y^b, x^c y^d)$$

respects the matrix product: if A and B are matrices with integer coefficients, then $A(B(x, y)) = (AB)(x, y)$. Therefore, if we denote $f\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ the action by a matrix A, f^n is the action by the matrix A^n. Therefore the growth of $||(f^n)^*||$ is given exactly by the growth of $||A^n||$ (for example with the maximal coefficient norm), which is $c\lambda^n$ where λ is the eigenvector with maximum modulus (or polynomial if some power of A is unipotent). Thus $\lambda_1(f) = \lambda$ where λ is the eigenvalue of A with maximum modulus.

b) *Can you prove that these maps have positive entropy if the matrix* $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ *has an eigenvalue greater than* 1*?*

Note that, since f is not regular, Gromov-Yomdin's theorem cannot be directly applied. However, we can restrict f to $\mathbb{T} = S^1 \times S^1 \subset \mathbb{C}^2$ ($S^1 \subset \mathbb{C}$ being the unit circle); $\mathbb{T}$ is a topological torus, and if we pass to linear coordinates f acts as $(x_1, x_2) \mapsto (ax_1 + bx_2, cx_3 + dx_4)$; we can check explicitly that this map has positive entropy (see exercise 1.c of lecture 1 for a trace), thus f itself has positive entropy.

Problem 4. a) *Describe two configurations of points in* $\mathbb{P}^2$ *which do not simply differ by a linear automorphism, but for which the corresponding blow-ups are isomorphic (hint: it might be easier to think of this in reverse – start with a rational surface, and describe two different ways to blow it down to* $\mathbb{P}^2$*).*

Let $\mathbf{p}$ be a configuration of six points p_1, p_2, p_3, p_4, p_5, and p_6 with p_1, p_2, and p_3 collinear and the other three points chosen general. Let X_p be the blow-up. It is not too hard to check that there is a unique (-2)-curve on X_p, given by the strict transform of the line through the first three points.

The idea is that we can blow down X_p in another way. Contract the strict transforms of the three lines between p_4, p_5, and p_6, together with the exceptional divisors over the points p_1, p_2, and p_3. This yields a new configuration $\mathbf{q}$ of six points. What's the image of the (-2)-curve in $\mathbb{P}^2$? It's now a conic passing through all 6 points, by a computation similar to the one from lecture.

It follows that the new configuration can not be projectively equivalent to the old one: for $\mathbf{p}$ there is no conic through all of the points, while for $\mathbf{q}$ there is one. On the other hand, for $\mathbf{p}$ there are three points on a line, which is not the case for $\mathbf{q}$.

b) *Prove that a very general configuration of n points in* $\mathbb{P}^2$ *(over* $\mathbb{C}$*) is "Cremona-general", in the sense that an arbitrary sequence of standard involutions centered at three-tuples among the points is well-defined.*

Problems with Cremona transformations being defined only arise when three of our points become collinear after some sequence of transformations, which means that there is a (-2)-curve on the blow-up $X_{\mathbf{p}}$. Hence to show that a general configuration is Cremona-general, it's enough to show that for very general $\mathbf{p}$, there are no (-2)-curves. It is important here that we're working over $\mathbb{C}$ (or at least some other uncountable field), which the problem should have specified.

The trick is to specialize the points: it's enough to prove this for a single configuration of points, and it then holds for a very general one. Let Γ be a smooth cubic in $\mathbb{P}^2$, and blow up n points on Γ. Observe that the strict transform of Γ on X is an anticanonical divisor.

If there is a (-2)-curve C on the blow-up, then it satisfies $\Gamma \cdot C = 0$, by adjunction. Writing $C \sim dH - \sum_{i=1}^n m_i E_i$, and noting that $E_i|_\Gamma$ is the point p_i in $\operatorname{Pic}^0(\Gamma)$, we obtain

$$dH|_\Gamma \sim \sum_{i=1}^n m_i p_i$$

in $\operatorname{Pic}^0(\Gamma)$. The set of configurations $(p_i)_{1\leq i\leq n}$ in E^n for which this holds with any particular choice of d and the m_i's is a proper Zariski closed subset of E^n. Since the set of all possible d and m_i is only countable, we conclude that there are no (-2)-curves on the blow-up as long as the p_i are very general (i.e. outside of some countable union of Zariski closed subsets of the configuration space. (That there actually exist such points outside of such a countable union requires that we work over an uncountable field.)

c) *Suppose that* $\mathbf{p}$ *is a very general configuration of* 10 *points in* $\mathbb{P}^2$. *Show that there exist infinitely many other configurations* $\mathbf{q}$ *such that no two* $\mathbf{p}$ *and* $\mathbf{q}$ *are projectively equivalent, but such that* $X_\mathbf{p} \cong X_\mathbf{q}$.

The argument here very closely follows [Les15, Lemma 3] (which is in dimension 3), so I will be a little brief. The idea is to prove that some special configuration $\mathbf{p}$ has infinite orbit; the result for a very general configuration then follows. Choose our initial configuration so that the blow-up X has three points on a line, but no other (-2)-curves; this is possible using the construction from the previous exercise. Let C be this (-2)-curve.

Now we use exactly the construction that appeared in the lecture. We repeatedly make a Cremona transformation at the last three points, and then reorder the points to move these three points to the beginning of the list. The growth of the degrees is as follows:

d	m_1	m_2	m_3	m_4	m_5	m_6	m_7	m_8	m_9	m_{10}
1	0	0	0	0	0	0	0	0	0	0
2	1	1	1	0	0	0	0	0	0	0
4	2	2	2	1	1	1	0	0	0	0
8	4	4	4	2	2	2	1	1	1	0
14	7	7	6	4	4	4	2	2	2	1

Remember that the Cremona involution gives isomorphism $X_\mathbf{p} \to X_\mathbf{q}$, so the surfaces we obtain are all isomorphic (the different degrees we measure for the image of C reflect the fact that we're blowing down different (-1)-curves, and so working in a different basis for $N^1(X)$.

Here the nth row is the class of the strict transform of C after the first $n-1$ Cremona transformations. For each configuration, there is a unique effective (-2)-curve class, but these classes all have different degrees. We conclude that these configurations are not projectively equivalent and so the Cremona orbit contains infinitely many nonisomorphic configurations of points.

One upshot of this problem is that it's not easy to define a moduli space of (non-minimal) rational surfaces in any reasonable way.

Problem 5. *Let $\phi_M : E \times E \to E \times E$ be a linear automorphism of a torus. Determine the k-periodic points. How does the number of periodic points grow with k?*

Let $E \cong \mathbb{C}/\Lambda$; finding the k-periodic points is the same as solving the equation $M^k v = v$ modulo Λ. Suppose first that ϕ_M has positive entropy (which is the same as asking that 1 is not an eigenvalue of M^k for any k). For each vector $w \in \Lambda \times \Lambda$, the equation $M^k v = v + w$ has exactly one solution $v \in \mathbb{C}^2$; however some of these solutions coincide in $E \times E$. Thus the number of k-periodic points in $E \times E$ is equal to the cardinality of Λ^2/Λ_k, where

$$\Lambda_k = \{w \in \Lambda^2 | (M^k - I)^{-1} w \in \Lambda^2\} = (M^k - I)\Lambda^2.$$

Call $A_k = M^k - I$; if we fix a basis v_1, v_2 of Λ, the matrix of the action of A_k on Λ^2 with basis $(v_1, 0), (0, v_1), (v_2, 0), (0, v_2)$ is

$$B_k = \begin{pmatrix} A_k & 0 \\ 0 & A_k \end{pmatrix}.$$

Since $\det B_k = (\det A_k)^2 \cong c\lambda^{2k}$ (λ being the maximum eigenvalue of M), the number of k-periodic points grows as $|\Lambda^2/\Lambda_k| = \det B_k \cong c\lambda^{2k} = c\lambda_1(\phi_M)$.
If M is unipotent, we have seen in lecture 1 that ϕ_M preserves an elliptic fibration $\pi \colon E \times E \to E'$, and that the action on the base is the identity. It is not difficult to see that ϕ_M acts as a translation on the fibres of π, and that the action is periodic on a countable number of fibres. Therefore, ϕ_M^k admits an infinite number of periodic points.

Problem 6. a) *Let $f \colon X \to X$ be a zero entropy automorphism. Show that we can define polynomial analogues of the dynamical degrees*

$$d_p(f) = \limsup_{n \to +\infty} \frac{\log \|(f^n)^*_p\|}{\log n} \in \mathbb{N}.$$

The eigenvalues of the $f_p^* \colon H^{2p}(X, \mathbb{C}) \to H^{2p}(X, \mathbb{C})$ are algebraic integers; since the entropy is zero, they have to have modulus ≤ 1, hence, by Kronecker's lemma they are roots of unity and for some iterate of f they are all equal to 1 (i.e. f^* is unipotent). In this case we can explicitly show that the coefficients grow polynomially, hence the polynomial dynamical degree exists (and it is equal to the maximal dimension of a Jordan block minus 1).

b) *($*$) Does this work for birational transformations?*

Nobody knows!

c) *Let* $\dim(X) = 3$*; show that* $d_1(f) \leq 4$.

Idea: work by contradiction, and fix a base $v_1, \dots v_k$ ($k = d_1(f) + 1 \geq 6$) for a maximal Jordan block for f_1^*. We can cut X with a hyperplane section and apply Hodge's index theorem to show that, if $v, w \in H^{1,1}(X)$ are non-collinear, then $v.v, v.w, w.w$ cannot be simultaneously 0. By computing $(f^n)^* v_k . (f^n)^* v_k$ and $(f^n)^* v_{k-1} . (f^n)^* v_{k-1}$, we find that $d_2(f) \geq 2d_1(f) - 3$; since $d_2(f) = d_1(f^{-1})$, the same proof shows that $d_2(f^{-1}) = d_1(f) \geq 2d_1(f^{-1}) - 3 \geq 4d_1(f) - 9$, contradiction.

d) *($*$) Prove or give a counterexample: suppose that $\phi : X \dashrightarrow X$ is a dominant rational map. Then $\lambda_i(\phi)$ is an algebraic integer.*

This is an open question. In the case that ϕ is a morphism, it is certainly true: the $\lambda_i(\phi)$ are eigenvalues of a matrix with integral entries, and hence certainly integral. In general, however, little is known.

References

[Dem97] Jean-Pierre Demailly, *Complex analytic and differential geometry*, Citeseer, 1997.

[Les15] John Lesieutre, *Derived-equivalent rational threefolds*, Int. Math. Res. Not. IMRN (2015), no. 15, 6011–6020.