Utah Summer School on Higher Dimensional Algebraic Geometry
Problem session #3: Counterexamples in birational geometry
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Problem 1. Let E be an elliptic curve; the Kummer surface X of E is the $K3$ surface obtained as the minimal resolution of the quotient $E \times E / -id$. Let $M \in SL_2(\mathbb{Z})$ be an infinite order semi-simple matrix.

a) Show that M induces an automorphism $\phi\colon X \to X$.

b) Show that the strict transform C of the curve $E \times \{0\}$ is a (-2)-curve in X.

c) Show that there exists an infinite number of (-2)-curves whose classes in $N^1(X)$ are distinct (hint: take C and all its iterates).

d) Let E be an elliptic curve with an order 6 automorphism. Show that $(E \times E)/\tau$ is a rational surface.

Problem 2. Fill in the details in the derived categories example:

a) Let $X_{\mathbf{p}}$ and $X_{\mathbf{q}}$ be the blow-ups of $\mathbb{P}^n$ $(n \geq 3)$ at two configurations of points. Show that $X_{\mathbf{p}}$ and $X_{\mathbf{q}}$ are isomorphic if and only if $\mathbf{p}$ and $\mathbf{q}$ differ only by the action of $\mathrm{PGL}(n+1)$ and permutations of the points.

b) Prove that a very general configuration of 8 or more points in $\mathbb{P}^3$ has infinite orbit under Cremona transformations (up to the action of $PGL(4)$)

Problem 3. Suppose that $\phi : X \to X$ is a positive entropy automorphism of a surface, and D is the leading eigenvector of the action of ϕ^* on $N^1(X)$. Suppose that C is a ϕ-periodic curve. Show that $D \cdot C = 0$. Can you prove the converse?

Problem 4. Let X be a smooth projective surface. If D is any pseudoeffective divisor, it admits aunique *Zariski decomposition* $D = P + N$, in which P is nef, N is effective, and $P \cdot N_i = 0$ for each component of N. One might try to generalize this to higher dimensional settings, but there is trouble...

a) Let X be the blow-up of $\mathbb{P}^2$ at 3 collinear points, and let $D = 3H - 2E_1 - 2E_2 - 2E_3$. What is the Zariski decomposition of D?

b) Let D be the divisor with $\mathbf{B}_{-}(D)$ not Zariski-closed, discussed in lecture. Show that D can not be expressed in the form $D = P + N$ with P nef and N effective.

c) Show that there does not even exist a birational model $\pi : Y \to X$ and a decomposition $\pi^* D = P + N$, with P nef and N effective.

Problem 5. a) Suppose that X is a surface and D is a nef class on X. Show that the number of curves with $D \cdot C = 0$ is either finite or uncountable.

b) Let $X = \mathrm{Bl}_8 \mathbb{P}^3$. Show that $-K_X$ is nef, but there exists an infinite discrete set of curves on X with $-K_X \cdot C = 0$.

Problem 6. Find a big $\mathbb{R}$-divisor with non-closed $\mathbf{B}_-(D)$.