Utah Summer School on Higher Dimensional Algebraic Geometry
Problem session #3: Counterexamples in birational geometry
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July 21, 2016

Problem 1. *Let E be an elliptic curve; the Kummer surface X of E is the $K3$ surface obtained as the minimal resolution of the quotient $E \times E/-id$. Let $M \in SL_2(\mathbb{Z})$ be an infinite order semi-simple matrix.*

a) *Show that M induces an automorphism $\phi \colon X \to X$.*
We have the following commuting diagram

$$\begin{array}{ccc} Y & \xrightarrow{\rho_1} & X \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi} \\ E \times E & \xrightarrow{\rho} & E \times E/-id \end{array}$$

where Y is the blow-up of $E \times E$ along the 16 two-torsion points in $E \times E$. Since the action of M on $E \times E$ preserves the two-torsion points, M induces an automorphism ψ_M on Y (the action on the exceptional divisors being given by the differential of ϕ_M); since $\psi_M(-p) = -\psi_M(p)$, this automorphism descends to an automorphism ϕ of X.

b) *Show that the strict transform C of the curve $E \times \{0\}$ is a (-2)-curve in X.*
Once we know that X is a $K3$ surface, we can just apply adjunction formula to get

$$K_C = (K_X + \mathcal{O}_X(C))|_C = \mathcal{O}_C(C)$$

and

$$C.C = \deg(\mathcal{O}_C(C)) = \deg K_C = -2$$

since $C \cong \mathbb{P}^1$.
One can also explicitly write ρ_1 in local coordinates around the exceptional divisors for π_1 to compute $C.C = -2$. In a similar fashion one can also show that X is actually a $K3$ surface: indeed, since ρ_1 is $2:1$ outside the exceptional divisors E_i for π_1 and $1:1$ on the E_i-s, when pulling back a meromorphic 4-form α on X to Y the E_i-s appear as simple zeroes together with the pull-back of zeroes of α; therefore,

$$K_Y = \sum E_i = \rho_1^* K_X + \sum E_i,$$

thus $K_X = 0$.

c) *Show that there exists an infinite number of (-2)-curves whose classes in $N^1(X)$ are distinct (hint: take C and all its iterates).*
Let us first show that all the images of C by ϕ are distinct: indeed, let E_0 be the exceptional divisor relative to the (image of the) point $(0,0)$. The curve C intersects E_0 only at the point p relative to the horizontal direction; since M is semi-simple of infinite order, the eigenspaces are not defined over $\mathbb{Q}$, so the orbit of p is infinite and contained into E_0. This shows that if $m \neq n$, then $\phi^m(C)$ and $\phi^n(C)$ intersect E_0 in two different points, thus in particular $\phi^m(C) \neq \phi^n(C)$. All the images of C are (-2)-curves since ϕ^* preserves the intersection product; since curves with negative self-intersection are fixed, their classes in $N^1(X)$ are distinct as well.

d) *Let E be an elliptic curve with an order 6 automorphism. Show that $(E \times E)/\tau$ is a rational surface.*

One way to do this is fairly direct, based on calculating the geometric invariants of X and applying Castelnuovo's criterion for detecting rational surfaces.

We take the indirect route. First, I claim that the surface X is uniruled. The quotient $(E\times E)/\tau$ has singularities of types $1/2(1,1)$, $1/3(1,1)$, and $1/6(1,1)$. Let X be its minimal resolution. Since $E \times E \to (E \times E)/\tau$ is étale in codimension 1, a computation of the discrepancies (i.e. the coefficients of the exceptional divisors in the resolution) gives $K_X \sim -\frac{1}{3}E_3 - \frac{2}{3}E_6$, where E_3 is the sum of the exceptional divisors over the singularities of type $1/3(1,1)$, and E_6 is the exceptional divisor over the unique singularity of type $1/6(1,1)$. In particular, $-K_X$ is effective, and this means that X is uniruled.

Notice as well that the surface X contains infinitely many (-1)-curves: arguing as in the first part of the problem, we find that the image of $E \times 0$ in the quotient is a (-1)-curve, and it has infinite order under the action of the automorphism group.

What uniruled surfaces are out there? According to the Enriques–Kodaira classification, they're all either rational, or blow-ups of $\mathbb{P}_C(\mathcal{E})$, where C is a curve of positive genus and $\mathcal{E}$ is a rank-2 vector bundle on E. The latter surfaces have only finitely many rational curves of negative self-intersection: any such curve must be contained in a fiber (since it can't dominate C), and only the finitely many reducible fibers can contain curves of negative self-intersection.

Problem 2. *Fill in the details in the derived categories example:*

a) *Let $X_{\mathbf{p}}$ and $X_{\mathbf{q}}$ be the blow-ups of $\mathbb{P}^n$ ($n \geq 3$) at two configurations of points. Show that $X_{\mathbf{p}}$ and $X_{\mathbf{q}}$ are isomorphic if and only if $\mathbf{p}$ and $\mathbf{q}$ differ only by the action of* $\mathrm{PGL}(n+1)$ *and permutations of the points.*

I am going to cheat here and refer you to [Les15]; this is Lemma 1. Here's the idea: if you blow up $\mathbb{P}^2$ at a bunch of points, there are different ways to blow it back down. For example, you can contract the strict transform of a line between two of the points you blew up. In higher dimensions, this is not the case: if you blow up points in $\mathbb{P}^3$, there are no contractible divisors on the blow-up except the ones you started with.

b) *Prove that a very general configuration of 8 or more points in $\mathbb{P}^3$ has infinite orbit under Cremona transformations (up to the action of $PGL(4)$)*

This is Lemma 3 in the same paper. The gist is that you should cheat a little bit: choose your original configuration so that the points are slightly special, with four of them coplanar. Then you can show that the new configurations you get after Cremona transformations are also somehow special, but in different ways (e.g. there is a cubic that's double at four of the eight points, and passes through the other four – a codimension 1 condition on configurations).

Problem 3. *Suppose that $\phi : X \to X$ is a positive entropy automorphism of a surface, and D is the leading eigenvector of the action of ϕ^* on $N^1(X)$. Suppose that C is a ϕ-periodic curve. Show that $D \cdot C = 0$. Can you prove the converse?*

Since the automorphism preserves the intersection form, we have

$$D \cdot \phi^n(C) = (\phi^*)^n(D) \cdot C = \lambda^n D \cdot C.$$

The term on the left and $D \cdot C$ are both rational, while λ^n is irrational. So it must be that $D \cdot C = 0$.

In fact the converse is true as well. I'm going to use exercise 5(a) below, though this can probably be avoided through judicious application of the Hodge index theorem. Suppose that $D \cdot C = 0$, but that C is not ϕ-periodic. We have seen that $D \cdot \phi^n(C) = 0$ for all n, giving an infinite set of curves with $D \cdot C = 0$. According to 5(a), it must be that all but finitely many of these curves fit into a positive-dimensional family; in particular, the class of C is nef. But then $D^2 = 0$, $D \cdot C = 0$, and $C^2 \geq 0$, contradicting the Hodge index theorem since D and C are nef. (Is this true?)

Problem 4. *Let X be a smooth projective surface. If D is any pseudoeffective divisor, it admits aunique* Zariski decomposition *$D = P + N$, in which P is nef, N is effective, and $P \cdot N_i = 0$ for each component of N. One might try to generalize this to higher dimensional settings, but there is trouble...*

a) *Let X be the blow-up of $\mathbb{P}^2$ at 3 collinear points, and let $D = 3H - 2E_1 - 2E_2 - 2E_3$. What is the Zariski decomposition of D?*

Note that the divisor D is actually big: it's $H + 2(H - E_1 - E_2 - E_3)$. The second summand is effective, and the first is the pullback of a big divisor from $\mathbb{P}^2$, hence big itself.

Now, let $N = \frac{3}{2}(H - E_1 - E_2 - E_3)$ and $P = \frac{1}{2}(3H - E_1 - E_2 - E_3)$. Then $D = P + N$, and we have $P \cdot N = 0$ as required. Since N is just 3/2 the class of the strict transform of a line through the points, it's surely effective, and so I just need to convince you that P is nef. Write $2P = (H - E_1) + (H - E_2) + (H - E_3)$, and notice that each of these is basepoint free, so P is nef as claimed.

b) *Let D be the divisor with $\mathbf{B}_-(D)$ not Zariski-closed, discussed in lecture. Show that D can not be expressed in the form $D = P + N$ with P nef and N effective.*

The D there was constructed as the leading eigenvector of some matrix $M_\sigma : N^1(X) \to N^1(X)$. In particular, D must be an extremal ray on the pseudoeffective cone $\overline{\text{Eff}}(X)$. If $D = P + N$, then P and N must both be proportional to D (since both of these are required to be pseudoeffective). But D is not nef, so it can't be proportional to P.

c) *Show that there does not even exist a birational model $\pi : Y \to X$ and a decomposition $\pi^* D = P + N$, with P nef and N effective.*

Sometimes, in higher dimensions, one can find an analog of the Zariski decomposition after passing to some higher birational model. For example, let X be the blow-up of $\mathbb{P}^3$ at two points, and let D be the strict transform of the plane between those two points. This divisor is not nef, since it has interscetion -1 with the strict transform of the line that goes through the two points. D can not be written as $P + N$ with P nef and N effective, by the same argument as in the previous part.

However, let $\pi : Y \to X$ be the blow-up along the line. Then $\pi^* D = P + N$, where N is the exceptional divisor of π, and D is the strict transform of the plane. This strict transform is now basepoint free, hence nef.

Sadly, this is not always to be. Let X and D be as in the problem. Then $D \cdot C = 0$ for an infinite set of curves. Let $\pi : Y \to X$ be any birational map from a smooth variety Y, and suppose that $\pi^* D = P + N$ where N is effective. I claim that P can't be nef. Indeed, pushing forward, we have $D = f_* P + f_* N$. Since D is extremal, both of these classes must be proportional to D. Since N is effective and $\mathbb{R}_{\geq 0} D$ has no effective representative, it must be that $f_* N = 0$, so that the divisor N is π-exceptional. Now $C \subset X$ be a curve with $D \cdot C < 0$ and which is not contained in the image of the exceptional divisors of Y (which is a finite union of curves, so we can certainly find such a C). Let $\tilde{C}$ be the strict transform of C on Y. Then $N \cdot \tilde{C} > 0$ since $\tilde{C}$ is not contained in the exceptional divisor. But this gives $P \cdot C = D \cdot C - N \cdot C < 0$, so P can't be nef.

Problem 5. a) *Suppose that X is a surface and D is a nef class on X. Show that the number of curves with $D \cdot C = 0$ is either finite or uncountable.*

I'm going to punt on this one as well: you can find this as Lemma 3.1 in [Tot09].

b) *Let $X = \mathrm{Bl}_8 \mathbb{P}^3$. Show that $-K_X$ is nef, but there exists an infinite discrete set of curves on X with $-K_X \cdot C = 0$.*

This is the main result of [LO16]. The construction is based on using Cremona transformations on the blow-up; it's pretty similar to the derived category example we discussed in lecture.

Problem 6. *Find a big $\mathbb{R}$-divisor with non-closed $\mathbf{B}_-(D)$.*

We saw in lecture that there exists a smooth threefold X and a pseudoeffective divisor D on X such that there is a Zariski dense set of curves C_n with $D \cdot C_n < 0$.

Now, let $Y = \mathbb{P}_X(\mathcal{O} \oplus \mathcal{O}(1))$, where $\mathcal{O}(1)$ is a very ample divisor on X. There are two maps on Y: first, there is $\pi : Y \to X$, the $\mathbb{P}^1$-bundle structure. Second, there is $f : Y \to CX$, the map to the cone over X obtained by contracting the negative section of Y. Let $i : X \to Y$ be the inclusion of the negative section, and set $D' = f^* H + \pi^* D$, where H is ample on CX.

Since H is ample, $f^* H$ is big and nef. Since D is pseudoeffective, so too is $\pi^* D$. It follows that the divisor D' is big. Moreover, $f^* H \cdot i(C_n) = 0$ since $i(C_n)$ is contracted by f. Thus $D' \cdot i(C_n) = D \cdot C_n < 0$, and so D' is also negative on a countably set of curves.

This means that each $i(C_n)$ is contained in $\mathbf{B}_-(D')$, but we still need to explain why this set is not Zariski closed. Since the $i(C_n)$ are dense in a divisor, it's sufficient to prove that $\mathbf{B}_-(D')$ does not contain any divisor. This follows from the fact that $f^* H$ and $\pi^* D$ are both movable classes, together with the results of Nakayama [Nak04].

References

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