Utah Summer School on Higher Dimensional Algebraic Geometry
Problem session #4: Invariant fibrations and classification problems
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Problem 1. Let $\psi : S \to S$ be a positive entropy automorphism of a rational surface, and let $\phi : S \times C \to S \times C$ be given by $\psi \times \mathrm{id}$, where C is a curve. Compute $\lambda_1(\phi)$ and $\lambda_2(\phi)$.

Problem 2. Suppose that $\dim X = 3$, $\phi : X \dashrightarrow X$ is birational, and $\pi : X \dashrightarrow Y$ is a map to a surface with $\pi \circ \phi = \psi \circ \pi$. Prove that $\lambda_1(\phi) = \lambda_2(\phi)$.

Problem 3. Let $f\colon X \dashrightarrow X$ be an automorphism of a surface with positive entropy. Show that the orbit of a very general point is Zariski-dense.

Problem 4. Suppose that X is a variety of general type (i.e. K_X is big). Show that $\mathrm{Bir}(X)$ is finite (you may assume that we are working over a field of characteristic 0).

Problem 5. Suppose that $\pi : X \to \mathbb{P}^3$ is the blow-up at some set of points. Show that X does not admit any automorphism of positive entropy. [Hint: let E be one of the exceptional divisors, and suppose it has infinite orbit under $\phi : X \to X$. Can the $\phi^n(E)$ intersect?]

Problem 6. Let $X = \mathrm{Hilb}^n(S)$ be the Hilbert scheme of points of a $K3$ surface (i.e. the minimal resolution of the symmetric product $\mathrm{Sym}^n(S)$. Then X is a $2n$-dimensional irreducible symplectic holomorphic (or hyperkähler) manifold, and a result by Verbitsky says that the cup-product induces injections

$$S^p(H^2(X,\mathbb{C})) \hookrightarrow H^{2p}(X,\mathbb{C})$$

for $p = 1, \ldots, n$.
Let $f\colon S \to S$ be a positive entropy automorphism with $\lambda_1(f) = \lambda > 1$. Show that f induces an automorphism of X and compute its dynamical degrees.

Problem 7. Let E be an elliptic curve with an order 6 automorphism τ.

a) Show that $(E \times E \times E)/\tau$ admits an imprimitive automorphism of positive entropy (hint: if the automorphism is induced by M, what will the dynamical degrees be? Find a matrix so that $\lambda_1(\phi) \neq \lambda_2(\phi)$.

b) For what values of n is E^n/τ uniruled? ($*$) Rational?

Problem 8. Consider a complete intersection in $\mathbb{P}^3 \times \mathbb{P}^3$ of general hypersurfaces of type $(1,1)$, $(1,1)$, $(2,2)$. Check that X is a Calabi-Yau threefold of Picard rank 2. Construct two involutive pseudoautomorphisms on X, and find a curve in the indeterminacy locus of each. Show that the composition has $\lambda_1 > 1$.

Problem 9. Suppose that $\phi : X \to X$ is a positive entropy automorphism of a smooth threefold, and that $E \subset X$ is a smooth divisor isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Show that for any nonzero n, $E \cap \phi^n(E)$ is a union of rulings of E.