Utah Summer School on Higher Dimensional Algebraic Geometry
Problem session #4: Invariant fibrations and classification problems
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Problem 1. *Let $\psi : S \to S$ be a positive entropy automorphism of a rational surface, and let $\phi : S \times C \to S \times C$ be given by $\psi \times \mathrm{id}$, where C is a curve. Compute $\lambda_1(\phi)$ and $\lambda_2(\phi)$.*

Let $\pi \colon S \times C \to C$ be the natural projection and let $\lambda_p(\phi|\pi)$ denote the relative dynamical degrees; since the action on the base C is the identity, we have $\lambda_p(\phi|\pi) = \lambda_p(\psi)$. By Dinh-Nguyen [DN11] we have

$$\lambda_1(\phi) = \max\{\lambda_1(id_C), \lambda_1(\phi|\pi)\} = \max\{1, \lambda_1(\psi)\} = \lambda_1(\psi)$$

and, using exercise 1 of lecture 2 and the Theorem on surfaces in lecture 1, we get

$$\lambda_2(\phi) = \lambda_1(\phi^{-1}) = \lambda_1(\psi^{-1}) = \lambda_1(\psi).$$

Problem 2. *Suppose that $\dim X = 3$, $\phi : X \dashrightarrow X$ is birational, and $\pi : X \dashrightarrow Y$ is a map to a surface with $\pi \circ \phi = \psi \circ \pi$. Prove that $\lambda_1(\phi) = \lambda_2(\phi)$.*

Since the fibres of π are 1-dimensional, $\lambda_0(\phi|\pi) = \lambda_1(\phi|\pi) = 1$. By Dinh-Sibony we have

$$\lambda_1(\phi) = \max\{\lambda_1(\psi), \lambda_1(\phi|\pi)\} = \lambda_1(\psi)$$

and

$$\lambda_2(\phi) = \lambda_1(\phi^{-1}) = \lambda_1(\psi^{-1}) = \lambda_1(\psi).$$

Problem 3. *Let $f \colon X \dashrightarrow X$ be an automorphism of a surface with positive entropy. Show that the orbit of a very general point is Zariski-dense.*

If this wasn't the case, then by Amerik-Campana [AC08], f would either be of finite order (contradiction), or preserve a fibration onto a curve, contradicting the Theorem on surfaces in lecture 1.

Problem 4. *Suppose that X is a variety of general type (i.e. K_X is big). Show that $\mathrm{Bir}(X)$ is finite (you may assume that we are working over a field of characteristic 0).*

This result is originally due to Matsumura. By definition, there's some m so that $|mK_X|$ gives a map $\phi_{|mK_X|} : X \to \mathbb{P}^N$ that is birational onto its image M. Then $\mathrm{Bir}(X) = \mathrm{Aut}(M)$, which is a subset of $\mathrm{PGL}(|mK_X|^*)$. The image is exactly the set of things the equations of the subvariety defining M, and so it's a closed algebraic subgroup of $\mathrm{PGL}(|mK_X|^*)$.

Either this subgroup is 0-dimensional, in which case we're done, or it's positive-dimensional, in which case we're going to get a contradiction with the fact that X is general type. If it's positive dimensional, it has a one-dimensional algebraic subgroup G, which must be either $\mathbb{G}_a$ or $\mathbb{G}_m$ (you can find a discussion of this in MO thread #143203).

Now, if x is any point on X, we can look at the orbit $G \cdot x$. This shows that there's a rational curve through every point on X, and in fact these curves belong to an algebraic family. By a standard Hilbert scheme argument, there's a dominant rational map $X' \to X$, where X' admits a map $X' \to Y$ with general fibers $\mathbb{P}^1$: here X' is a component of the universal family over the Hilbert scheme, and Y is the component of the Hilbert scheme parametrizing our rationals curves on X. The map $X' \to X$ is separable, since we're in characteristic 0, and this means that if we pullback a nonzero section of mK_X, we get a nonzero section of $mK_{X'}$. But by repeated application of adjunction, this would yield a holomorphic 1-form on $\mathbb{P}^1$, which is impossible.

Problem 5. *Suppose that $\pi : X \to \mathbb{P}^3$ is the blow-up at some set of points. Show that X does not admit any automorphism of positive entropy. [Hint: let E be one of the exceptional divisors, and suppose it has infinite orbit under $\phi : X \to X$. Can the $\phi^n(E)$ intersect?]*

We will actually show that X can't have any automorphisms of infinite order, except possibly the lifts of linear maps from $\mathbb{P}^3$, which have entropy 0. Suppose that $\phi : X \to X$ is an infinite order automorphism. Let E be one of the exceptional divisors of the automorphism. Either E is ϕ-periodic, or E has infinite orbit under ϕ.

If E is ϕ-periodic, then we can replace ϕ by some iterate and assume that E is fixed; since ϕ has infinite order, this iterate is not the identity map. Now, let $X \to Y$ be the blow-down of E. Since E is ϕ-invariant, the map ϕ descends to an automorphism of Y. In this case, we replace X with Y and start from the beginning.

As a result, we may suppose that E is not periodic under ϕ, so that the divisors $\phi^m(E)$ are all distinct.

Problem 6. *Let $X = \mathrm{Hilb}^n(S)$ be the Hilbert scheme of points of a $K3$ surface (i.e. the minimal resolution of the symmetric product $\mathrm{Sym}^n(S)$. Then X is a $2n$-dimensional irreducible symplectic holomorphic (or hyperkähler) manifold, and a result by Verbitsky says that the cup-product induces injections*

$$S^p(H^2(X,\mathbb{C})) \hookrightarrow H^{2p}(X,\mathbb{C})$$

for $p = 1, \ldots, n$.
Let $f\colon S \to S$ be a positive entropy automorphism with $\lambda_1(f) = \lambda > 1$. Show that f induces an automorphism of X and compute its dynamical degrees.

The product $f \times f \times \cdots \times f$ (n times) clearly induces an isomorphism of $Sym^n(S)$; the Hilbert scheme is obtained by resolving $Sym^n(S)$ along the singular set, which is the image of the set $\Delta = \{(x_1, \ldots, x_n) \in S \times \cdots \times S | x_i = x_j \text{ for some } i \neq j\}$. Since Δ is preserved by $f \times f \times \cdots \times f$, we have an induced automorphism g of $\mathrm{Hilb}^n(S)$.
Since the natural projection $S^n \dashrightarrow \mathrm{Hilb}^n(X)$ is generically finite, by exercise 1 of lecture 2 we have

$$\lambda_p(g) = \lambda_p(f \times \cdots \times f)$$

and in particular $h_{\text{top}}(g) = h_{\text{top}}(f \times \cdots \times f)$.
It is not hard to see that

$$h_{\text{top}}(f \times \cdots \times f) = n h_{\text{top}}(f).$$

Indeed, two points $(x_1, \ldots, x_n)$ and $(y_1, \ldots y_n)$ define (N, ϵ)-separated orbits if there exists i such that x_i and y_i define (N, ϵ)-separated orbits, and conversely, if $\mathbf{x}$ and $\mathbf{y}$ define (N, ϵ)-separated orbits, then for some i, x_i and y_i define $(N, \epsilon/\sqrt{n})$- separated orbits. This allows to give a precise estimate of the number of (N, ϵ)-separated orbits for $f \times \cdots \times f$.
Now, by Verbitsky's result we have $\lambda_p(g) = \lambda_1(g)^p$ for $p = 1, \ldots, n$: indeed, if $v \in Pic(X) \subset H^2(X, \mathbb{Z})$ is an eigenvector for g^* with maximal eigenvalue $\lambda_1(g)$, then $v^p \in H^{2p}(X, \mathbb{Z})$ is a (non-trivial) eigenvector for g^* with eigenvalue $\lambda_1(g)^p$, so that $\lambda_p(g) \geq \lambda_1(g)^p$; the converse inequality follows from log-concavity. Furthermore, the same proof applied to g^{-1} shows that $\lambda_{2n-p}(g) = \lambda_1(g)^p$ for $p = 1, \ldots, n$.
In conclusion, we have

$$n h_{\text{top}}(f) = h_{\text{top}}(g) = \log(\lambda_n(g)) = n \log \lambda_1(g)$$

so that $\lambda_1(f) = \lambda_1(g)$ and

$$\lambda_p(g) = \begin{cases} \lambda_1(f)^p & p = 1, \ldots, n \\ \lambda_1(f)^{2n-p} & p = n+1, \ldots, 2n \end{cases}$$

Problem 7. *Let E be an elliptic curve with an order 6 automorphism τ.*

a) *Show that $(E \times E \times E)/\tau$ admits an imprimitive automorphism of positive entropy (hint: if the automorphism is induced by M, what will the dynamical degrees be? Find a matrix so that $\lambda_1(\phi) \neq \lambda_2(\phi)$.*

For this one, I'll refer you to [OT15], where you can find it as Lemma 4.3. The paper has a nice discussion of using relative dynamical degrees to show that there is no invariant fibration.

b) *For what values of n is E^n/τ uniruled? (*) Rational?*

There's a good way to check whether something like this is uniruled, using the BDPP theorem: we need to find a model $\pi : X \to E^n/\tau$ with terminal singularities, and then try to figure out whether the canonical class is pseudoeffective or not. In particular, if we can show that the $-K_X$ is effective, then the variety is uniruled. On the other hand, if K_X is effective, then it isn't.

We still have to find X and compute the canonical class, however. The singular points on the quotient are all isolated, and they come from the fixed points of the action of τ on E^n. Our variety has three different kinds of singularities, depending on the stabilizers of the point: they can have stabilizer of order 2, 3, or 6.

We need to make a standard calculation of how the canonical class changes when we resolve these singularities by blowing up (a good reference for this sort of thing is [Rei87]). Let ω be a dth root of unity acting on $\mathbb{C}[x, y, z]$ by multiplication by ω in each variable (we'll take $d = 2, 3, 6$, depending on which point we want). The quotient is (affine locally) Spec of the ring of invariants, which is generated by monomials $x^i y^j z^k$ with $i + j + k = d$. This is isomorphic to the cone over the dth Veronese embedding $\mathbb{P}^{n-1} \to \mathbb{P}^N$.

Let P be the cone in question. Let $\pi : C = \mathbb{P}_{\mathbb{P}^{n-1}}(\mathcal{O} \oplus \mathcal{O}(d)) \to P$ be resolution obtained by blowing up the cone point, with exceptional divisor $E \cong \mathbb{P}^{n-1}$ that has normal bundle $\mathcal{O}(-(n-1))$. Then $K_C = \pi^* K_P + aE$ for some a. By adjunction, $K_E = (a+1)E|_E$, and we have seen that $K_E = \mathcal{O}_{\mathbb{P}^2}(-3)$ and $E|_E = \mathcal{O}_{\mathbb{P}^2}(-d)$. Putting this together, we obtain $a = \frac{n}{d} - 1$. When $d = 2$ this is positive as long as $n \geq 3$, so the singularity in question is terminal. We don't need to blow it up to obtain our X. When $d = 3$ this is non-negative for $n = 3$ and positive for $n \geq 4$. The worst singularities come from $d = 6$. Once $n \geq 6$, the discrepancy is non-negative, and K_X is effective, so the variety is of Calabi–Yau type. For $3 \leq n \leq 5$, after blowing up the non-terminal singularities, we obtain a terminal variety on which $-K_X$ is effective. This shows that X is uniruled.

The question of which of these varieties are rational has been an active topic lately. For $n = 3$ this is proved in [OT15]. For $n = 4$ it's known to be unirational [COV].

Problem 8. *Consider a complete intersection in $\mathbb{P}^3 \times \mathbb{P}^3$ of general hypersurfaces of type $(1,1)$, $(1,1)$, $(2,2)$. Check that X is a Calabi-Yau threefold of Picard rank 2. Construct two involutive pseudoautomorphisms on X, and find a curve in the indeterminacy locus of each. Show that the composition has $\lambda_1 > 1$.*

This example is from [Ogu14], where you can find a detailed description of many of the nice aspects of the geometry of this variety. This example appears as Example 6.1.

Problem 9. *Suppose that $\phi : X \to X$ is a positive entropy automorphism of a smooth threefold, and that $E \subset X$ is a smooth divisor isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Show that for any nonzero n, $E \cap \phi^n(E)$ is a union of rulings of E.*

Let D be a dominant eigenvector of the pullback map $\phi^* : N^1(X) \to N^1(X)$ (i.e. an eigenvector with the largest eigenvalue). We saw during the lectures that this eigenvalue is a real number, which is greater than 1 by hypothesis. In fact, it must also be irrational: the matrix for ϕ^* has integer entries, as does its inverse. As a result, the only possible rational roots of the characteristic polynomial are -1 and 1.

Now, we compute a certain three-way intersection in two different ways.

$$\begin{aligned} D \cdot E \cdot \phi^n(E) &= (D \cdot \phi^n(E))|_E \\ &= \phi^{n*} D \cdot \phi^{n*} E \cdot \phi^{n*}(\phi^n(E)) \\ &= \lambda^n D \cdot \phi^{-n}(E) \cdot E \\ &= \lambda^n (D \cdot \phi^{-n}(E))|_E \end{aligned}$$

The quantity on the right of the first row is a rational number, since it's the intersection of two Cartier divisors on a smooth surface. The quantity on the right of the bottom is λ^n times a rational number. The only possibility is that $D \cdot E \cdot \phi(E) = 0$.

Now, D is nef, and so $D \cdot E$ is also nef. A Hodge index theorem argument shows that $D \cdot E$ is nonzero. Hence $D|E$ must be (numerically) a multiple of one of the rulings on E. The fact that $D \cdot E \cdot \phi^n(E) = 0$ then implies that $\phi^n(E)$ must be (again numerically) a multiple of the same ruling, for every value of n. But the only curve classes numerically equivalent to a ruling are rulings, and so $E \cdot \phi^n(E)$ must be a union of rulings on E for every value of n.

References

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