Utah Summer School on Higher Dimensional Algebraic Geometry
Problem session #1: Dynamical degrees & entropy
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Problem 1. a) Let $X = \mathbb{P}^1$ and let $\phi : X \to X$ be the map $z \mapsto z^d$. Compute the entropy $h_{\text{top}}(\phi)$ in two different ways. Do your answers agree?

i) Directly from the topological definition;

ii) Using the Gromov-Yomdin theorem.

b) Show that no orbit of ϕ is dense for the usual topology, but that there exist points whose orbit is dense in S^1, and therefore Zariski-dense in $\mathbb{P}^1$.

c) Let E be an elliptic curve, and let M be an element of $\mathrm{SL}_2(\mathbb{Z})$ with $\phi_M : E \times E \to E \times E$ the induced automorphism. Repeat part (a) for this map. (You can try the example $M = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$).

Problem 2. Show that $\mathbb{P}^1 \times \mathbb{P}^1$ does not admit an automorphism of positive entropy. Can you prove this for any other classes of varieties?

In dimension 2, the dynamical degrees detect the existence of invariant fibrations for a birational map, according to the following theorem of Diller and Favre.

Theorem 1. Let X be a projective surface and let $f : X \to X$ be an automorphism. Then one of the following holds:

1. f^* is of finite order; in this case some iterate of f is isotopic to the identity (i.e. some $f^{\circ k}$ lies in $\mathrm{Aut}^0(X)$),
2. some iterate of f^* is unipotent of infinite order; in this case $\lambda_1(f) = 1$ and f preserves a fibration $\pi : X \to C$ onto a curve.
3. f^* is semi-simple; in this case $\lambda_1(f)$ is a Salem number (i.e. an algebraic integer whose conjugates over $\mathbb{Q}$ are $1/\lambda_1(f)$ and some complex numbers of modulus 1) and f does not preserve any fibration.

We won't give a full proof, but the next few problems demonstrate some aspects of this fact.

Problem 3. a) Suppose that $f : X \to X$ is an automorphism of a smooth projective surface, and there is a fibration to a curve $\pi : X \to C$ and an automorphism $g : C \to C$ with $\pi \circ f = g \circ \pi$. Show that $\lambda_1(f) = 0$.

b) Let X be the blow-up of $\mathbb{P}^2$ at the base locus of a pencil of cubics. Show that every automorphism of X must preserve the resulting elliptic fibration, and so has entropy 0.

Problem 4. Let X, f be as in Theorem 1 and let

$$\mathcal{C} = \{D \in N^1_{\mathbb{R}}(X), D.D \geq 0\}$$

be the positive cone for the intersection product.

a) Show that f^* preserves a line in $\mathcal{C}$.

b) Show that, if f^* preserves a line in the interior of $\mathcal{C}$, then f^* has finite order.

c) Show that if f^* preserves a single line in $\partial\mathcal{C}$, then some iterate of f^* is unipotent of infinite order, and that then $||(f^n)^*||$ grows as cn^2.

d) Show that if f^* preserves at least two lines in $\partial\mathcal{C}$ and no line in the interior of $\mathcal{C}$, then f^* is semi-simple and $||(f^n)^*||$ grows as $c\lambda^n$, where $\lambda = \lambda_1(f)$ is a Salem number. Furthermore, f^* preserves exactly two lines in $\mathcal{C}$, which are not defined over $\mathbb{Q}$.

Problem 5. Let's show that if $\phi : X \to X$ is an automorphism with unbounded degree, then it preserves an elliptic fibration. You might want to assume X is a K3 surface the first time through.

a) Show that there exists an integral nef class D with $\phi^*D = D$, $D^2 = 0$, $D \cdot K_X = 0$.

b) Show that $h^0(nD) \geq 2$ for sufficiently large n.

c) Show that the rational map determined by nD is an elliptic fibration.

Problem 6. Let E be an elliptic curve, and let M be an element of $\mathrm{SL}_2(\mathbb{Z})$ with $\phi_M : E \times E \to E \times E$ the induced automorphism. Describe the induced linear automorphism f^* and check the results of Theorem 1. In the semi-simple case, show that ϕ_M preserves a pair of smooth foliations $\mathcal{F}_+, \mathcal{F}_-$ whose leaves are dense in $E \times E$.

Problem 7. Fix a smooth plane cubic $E \subset \mathbb{P}^2$, and let p be a general point on E. We may define a rational map $\tau_p : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ as follows. Given a point $x \in \mathbb{P}^2$, draw the line L from x to p. Generically, this line meets E in two points. There is a unique involution of L that fixes these two points (why?). Let $\tau_p(x)$ be the image of x under this involution.

a) The map τ_p is a rational map, not an automorphism. Determine the indeterminacy locus of τ_p. Show that after blowing up the indeterminacy locus, we obtain a map $\tilde{\tau}_p : X_p \to X_p$ which is an involutive automorphism of a rational surface (or just trust us and go to (b)).

b) Compute the action of $\tilde{\tau}_p : X_p \to X_p$ on $N^1(X_p)$.

c) Show that if $q \in E$ is another point, there exists a rational surface X_{pq} such that τ_p and τ_q both lift to automorphisms of X_{pq}. Compute the matrices for the action of these involutions on $N^1(X_{pq})$.

d) Check that $\tau_p \circ \tau_q$ acts with infinite order on $N^1(X_{pq})$, but this map has entropy 0.

e) This means that $\tau_p \circ \tau_q$ preserves an elliptic fibration: can you find it?

f) Fix a third point r on E. Using the same construction, exhibit a positive entropy automorphism of a rational surface X_{pqr}.

Problem 8. a) Construct a compact metric space X and a map $\phi : X \to X$ with $h_{\text{top}}(\phi) = \infty$. Can you find such a map when $X = [0, 1]$?

b) Show (from the definition) that if $\phi : X \to X$ is an automorphism of a variety, $h_{\text{top}}(\phi)$ is finite. (Hint: in fact, show that if $\phi : X \to X$ is Lipschitz with constant C on a manifold X, then $h_{\text{top}}(\phi) \leq C \dim X$.)

Utah Summer School on Higher Dimensional Algebraic Geometry
Problem session #2: More dynamical degrees & examples
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Problem 1. a) Prove that if $\phi : X \dashrightarrow X$ is a birational transformation and $n = \dim(X)$, then $\lambda_{n-d}(\phi) = \lambda_d(\phi^{-1})$.

b) Let $f\colon X \dashrightarrow X$ and $g\colon Y \dashrightarrow Y$ be (bi)rational maps and let $\pi\colon X \dashrightarrow Y$ be a generically finite map such that $\pi \circ f = g \circ \pi$. Show that $\lambda_p(f) = \lambda_p(g)$ for all p.

c) Suppose that $\phi : X \to X$ is a positive entropy automorphism of a smooth threefold. Let D be a leading eigenvector of $\phi^* : N^1(X) \to N^1(X)$, and D' a leading eigenvector of $(\phi^{-1})^* : N^1(X) \to N^1(X)$. Show that either $D^2 = 0$ or $(D')^2 = 0$ (as elements of $N^2(X)$, or $H^{2,2}(X)$). (Hint: you can assume $\lambda_1(f)$ is a real eigenvalue)

Problem 2. Suppose that $D \subset \mathbb{P}^3$ is a surface of degree d, with multiplicities m_1, m_2, m_3, and m_4 at the four coordinate points. Compute the degree and multiplicities of $\mathrm{Cr}(D)$, where Cr is the standard Cremona involution. What if D is a curve instead of a surface?

Problem 3. a) Compute the dynamical degrees for the following affine maps: $(x, y) \mapsto (x^2y, xy)$, $(x, y) \mapsto (xy, y)$. What about the maps $(x, y) \mapsto (x^a y^b, x^c y^d)$ more generally?

b) Can you prove that these maps have positive entropy if the matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ has an eigenvalue greater than 1?

Problem 4. a) Describe two configurations of points in $\mathbb{P}^2$ which do not simply differ by a linear automorphism, but for which the corresponding blow-ups are isomorphic (hint: it might be easier to think of this in reverse – start with a rational surface, and describe two different ways to blow it down to $\mathbb{P}^2$).

b) Prove that a very general configuration of n points in $\mathbb{P}^2$ (over $\mathbb{C}$) is "Cremona-general", in the sense that an arbitrary sequence of standard involutions centered at three-tuples among the points is well-defined.

c) Suppose that $\mathbf{p}$ is a very general configuration of 10 points in $\mathbb{P}^2$. Show that there exist infinitely many other configurations $\mathbf{q}$ such that no two $\mathbf{p}$ and $\mathbf{q}$ are projectively equivalent, but such that $X_{\mathbf{p}} \cong X_{\mathbf{q}}$.

Problem 5. Let $\phi_M : E \times E \to E \times E$ be a linear automorphism of a torus. Determine the k-periodic points. How does the number of periodic points grow with k?

Problem 6. a) Let $f\colon X \to X$ be a zero entropy automorphism. Show that we can define polynomial analogues of the dynamical degrees

$$d_p(f) = \limsup_{n\to+\infty} \frac{\log\|(f^n)^*_p\|}{\log n} \in \mathbb{N}.$$

b) $(*)$ Does this work for birational transformations?

c) Let $\dim(X) = 3$; show that $d_1(f) \le 4$.

d) $(*)$ Prove or give a counterexample: suppose that $\phi : X \dashrightarrow X$ is a dominant rational map. Then $\lambda_i(\phi)$ is an algebraic integer.

Utah Summer School on Higher Dimensional Algebraic Geometry
Problem session #3: Counterexamples in birational geometry
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Problem 1. Let E be an elliptic curve; the Kummer surface X of E is the $K3$ surface obtained as the minimal resolution of the quotient $E \times E/-id$. Let $M \in SL_2(\mathbb{Z})$ be an infinite order semi-simple matrix.

a) Show that M induces an automorphism $\phi \colon X \to X$.

b) Show that the strict transform C of the curve $E \times \{0\}$ is a (-2)-curve in X.

c) Show that there exists an infinite number of (-2)-curves whose classes in $N^1(X)$ are distinct (hint: take C and all its iterates).

d) Let E be an elliptic curve with an order 6 automorphism. Show that $(E \times E)/\tau$ is a rational surface.

Problem 2. Fill in the details in the derived categories example:

a) Let $X_{\mathbf{p}}$ and $X_{\mathbf{q}}$ be the blow-ups of $\mathbb{P}^n$ $(n \geq 3)$ at two configurations of points. Show that $X_{\mathbf{p}}$ and $X_{\mathbf{q}}$ are isomorphic if and only if $\mathbf{p}$ and $\mathbf{q}$ differ only by the action of $\mathrm{PGL}(n+1)$ and permutations of the points.

b) Prove that a very general configuration of 8 or more points in $\mathbb{P}^3$ has infinite orbit under Cremona transformations (up to the action of $PGL(4)$)

Problem 3. Suppose that $\phi : X \to X$ is a positive entropy automorphism of a surface, and D is the leading eigenvector of the action of ϕ^* on $N^1(X)$. Suppose that C is a ϕ-periodic curve. Show that $D \cdot C = 0$. Can you prove the converse?

Problem 4. Let X be a smooth projective surface. If D is any pseudoeffective divisor, it admits aunique *Zariski decomposition* $D = P + N$, in which P is nef, N is effective, and $P \cdot N_i = 0$ for each component of N. One might try to generalize this to higher dimensional settings, but there is trouble...

a) Let X be the blow-up of $\mathbb{P}^2$ at 3 collinear points, and let $D = 3H - 2E_1 - 2E_2 - 2E_3$. What is the Zariski decomposition of D?

b) Let D be the divisor with $\mathbf{B}_-(D)$ not Zariski-closed, discussed in lecture. Show that D can not be expressed in the form $D = P + N$ with P nef and N effective.

c) Show that there does not even exist a birational model $\pi : Y \to X$ and a decomposition $\pi^* D = P + N$, with P nef and N effective.

Problem 5. a) Suppose that X is a surface and D is a nef class on X. Show that the number of curves with $D \cdot C = 0$ is either finite or uncountable.

b) Let $X = \mathrm{Bl}_8 \mathbb{P}^3$. Show that $-K_X$ is nef, but there exists an infinite discrete set of curves on X with $-K_X \cdot C = 0$.

Problem 6. Find a big $\mathbb{R}$-divisor with non-closed $\mathbf{B}_{-}(D)$.

Utah Summer School on Higher Dimensional Algebraic Geometry
Problem session #4: Invariant fibrations and classification problems
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Problem 1. Let $\psi : S \to S$ be a positive entropy automorphism of a rational surface, and let $\phi : S \times C \to S \times C$ be given by $\psi \times \mathrm{id}$, where C is a curve. Compute $\lambda_1(\phi)$ and $\lambda_2(\phi)$.

Problem 2. Suppose that $\dim X = 3$, $\phi : X \dashrightarrow X$ is birational, and $\pi : X \dashrightarrow Y$ is a map to a surface with $\pi \circ \phi = \psi \circ \pi$. Prove that $\lambda_1(\phi) = \lambda_2(\phi)$.

Problem 3. Let $f \colon X \dashrightarrow X$ be an automorphism of a surface with positive entropy. Show that the orbit of a very general point is Zariski-dense.

Problem 4. Suppose that X is a variety of general type (i.e. K_X is big). Show that $\mathrm{Bir}(X)$ is finite (you may assume that we are working over a field of characteristic 0).

Problem 5. Suppose that $\pi : X \to \mathbb{P}^3$ is the blow-up at some set of points. Show that X does not admit any automorphism of positive entropy. [Hint: let E be one of the exceptional divisors, and suppose it has infinite orbit under $\phi : X \to X$. Can the $\phi^n(E)$ intersect?]

Problem 6. Let $X = \mathrm{Hilb}^n(S)$ be the Hilbert scheme of points of a $K3$ surface (i.e. the minimal resolution of the symmetric product $\mathrm{Sym}^n(S)$. Then X is a $2n$-dimensional irreducible symplectic holomorphic (or hyperkähler) manifold, and a result by Verbitsky says that the cup-product induces injections

$$S^p(H^2(X, \mathbb{C})) \hookrightarrow H^{2p}(X, \mathbb{C})$$

for $p = 1, \ldots, n$.
Let $f \colon S \to S$ be a positive entropy automorphism with $\lambda_1(f) = \lambda > 1$. Show that f induces an automorphism of X and compute its dynamical degrees.

Problem 7. Let E be an elliptic curve with an order 6 automorphism τ.

a) Show that $(E \times E \times E)/\tau$ admits an imprimitive automorphism of positive entropy (hint: if the automorphism is induced by M, what will the dynamical degrees be? Find a matrix so that $\lambda_1(\phi) \neq \lambda_2(\phi)$.

b) For what values of n is E^n/τ uniruled? (*) Rational?

Problem 8. Consider a complete intersection in $\mathbb{P}^3 \times \mathbb{P}^3$ of general hypersurfaces of type $(1,1)$, $(1,1)$, $(2,2)$. Check that X is a Calabi-Yau threefold of Picard rank 2. Construct two involutive pseudoautomorphisms on X, and find a curve in the indeterminacy locus of each. Show that the composition has $\lambda_1 > 1$.

Problem 9. Suppose that $\phi : X \to X$ is a positive entropy automorphism of a smooth threefold, and that $E \subset X$ is a smooth divisor isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Show that for any nonzero n, $E \cap \phi^n(E)$ is a union of rulings of E.