Problems for M 11/2:

5.4.11 Let $B$ be the basis given by

$$b_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \quad b_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$ 

Find the $B$-matrix for the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $x \mapsto Ax$, where

$$A = \begin{bmatrix} 3 & 4 \\ -1 & -1 \end{bmatrix}.$$ 

(This just means the matrix for the transformation $T$, but where we use the basis $B$ on both sides.)

5.4.13 Consider the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$A = \begin{bmatrix} 0 & 1 \\ -3 & 4 \end{bmatrix}.$$ 

Find a basis $B$ with respect to which the transformation is diagonal.

1. Early in the course I mentioned that linear algebra gives you a way to find a formula for the Fibonacci numbers. I didn’t get to it in lecture, so I will let you work it out. The Fibonacci numbers are a sequence defined by $F_1 = 1$ and $F_2 = 1$, and $F_{n+1} = F_n + F_{n-1}$. They start off $1, 1, 2, 3, 5, 8, 13, \ldots$. You can read lots of fun facts about them on Wikipedia.

(a) Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, and $x$ be the vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Compute $Ax, A^2x, A^3x$. Convince yourself that $A^n x$ is the vector $\begin{bmatrix} F_{n+2} \\ F_{n+1} \end{bmatrix}$.

(b) Computing $A^n x$ directly is hard, but we can do it by working with coordinates in an eigenbasis. First, find the eigenvectors and eigenvalues of $A$ (hint: your answer will be a little messy, involving some $\sqrt{5}$’s.) Then write down a diagonalization of $A$.

(c) Let $B$ be the basis given by the eigenvectors you found. Compute the coordinate vector $[x]_B$.

(d) Our formula for linear transformations in an eigenbasis tells us that $[A^n(x)]_B = D^n [x]_B$, where $D$ is the diagonalization of $A$. Use your answers to the previous questions to find $[A^n(x)]_B$. 
(e) Convert this back into regular coordinates to get an expression for $A^n(x)$. What is your formula for $F_{n+1}$?

**Problems for W 11/4:**

5.4.18 Define $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ by $T(x) = Ax$, where $A$ is a $3 \times 3$ matrix with eigenvalues 5 and $-2$. Does there exist a basis $B$ such that the $B$-matrix for $T$ is a diagonal matrix? Discuss.

1. Find the roots of $x^2 - 4x + 13 = 0$.
2. Find $(3 + 4i)(2 - 6i)$.
3. Find $\frac{3+4i}{2-6i}$.
4. Write $1 + i$ in polar form, $re^{i\theta}$. Use this to compute $(1 + i)^5$.

**Problems for F 11/6:**

5.5.1 Find the (possibly complex) eigenvalues and eigenvectors for the matrix

$$A = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}. $$

5.5.4 Ditto, with

$$A = \begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix}. $$

5.5.9 Find the eigenvalues of $A$. The transformation determined by $A$ is a composition of a rotation and a scaling; give the angle of the rotation, and the scaling factor. (Hint: look at Example 6).

$$A = \begin{bmatrix} -\sqrt{3}/2 & 1/2 \\ -1/2 & -\sqrt{3}/2 \end{bmatrix}. $$

5.5.13 Consider the matrix

$$A = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}. $$

Find an invertible matrix $P$ and a matrix $C$ of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ (both with real entries) such that $A = PCP^{-1}$. You might find your answer to the first question useful.