18.02 Recitation
Problems
14 September 2011

A few questions from the problem sheet aren't included – email me if you want any additional solutions.

1. (1D-1) *Compute the cross products* $\langle 1,-2,1\rangle \times \langle 2,-1,-1\rangle$ *and* $\langle 2,0,-3\rangle \times \langle 1,1,-1\rangle$.

   We the formula with the determinant:

$$\begin{aligned}\langle 1,-2,1\rangle \times \langle 2,-1,-1\rangle &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & -2 & 1 \\ 2 & -1 & -1 \end{vmatrix} \\ &= (2-(-1))\hat{\imath} + (2-(-1))\hat{\jmath} + (-1-(-4))\hat{k} = \langle 3,3,3\rangle \\ \langle 2,0,-3\rangle \times \langle 1,1,-1\rangle &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 0 & -3 \\ 1 & 1 & -1 \end{vmatrix} \\ &= (0-(-3))\hat{\imath} + (-3-(-2))\hat{\jmath} + (2-0)\hat{k} = \langle 3,-1,2\rangle\,. \end{aligned}$$

2. *What is the volume of the parallelpiped with vertices at* $\langle 0,0,1\rangle$, $\langle a_1,a_2,0\rangle$, $\langle b_1,b_2,0\rangle$*? Is this consistent with our formula for the area of a triangle?*

   We can compute the volume of a parallelpiped using the scalar triple product (discussed in the textbook). It's

$$\langle 0,0,1\rangle \cdot (\langle a_1,a_2,0\rangle \times \langle b_1,b_2,0\rangle) = \begin{vmatrix} 0 & 0 & 1 \\ a_1 & a_2 & 0 \\ b_1 & b_2 & 0 \end{vmatrix} = (a_1b_2 - a_2b_1).$$

   The area of a triangle with legs $\vec{a}$ and $\vec{b}$ was $(a_1b_2 - a_2b_1)/2$, and so the area of the parallelogram spanned by these two vectors is $a_1b_2 - a_2b_1$. The parallelpiped in question has this parallelogram as its base, and height 1 (from the $\langle 0,0,1\rangle$), so this is consistent with the earlier computation.

3. *Consider the matrix* $M = \left(\begin{smallmatrix} 1 & 1 \\ 0 & -1 \end{smallmatrix}\right)$. *What is* M *applied to* $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$*? Can you interpret this transformation geometrically?* This sends the vector $\langle x,y\rangle$ to

$$\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ -y \end{pmatrix}$$

   Think about what this does to some specific vectors: if $y = 0$, then nothing changes at all. If y is positive, the point gets shifted to the right by an amount proportional to y (so it goes further as y gets larger), and then flipped over the x-axis. The first part of this is called a "shear": look at `http://en.wikipedia.org/wiki/Shear_mapping` for a good picture.

4. *What is the inverse of the matrix M above?*

Use the formula for the inverse of a 2×2 matrix.

$$M^{-1} = \frac{1}{|M|} \begin{pmatrix} m_{22} & -m_{12} \\ -m_{21} & m_{22} \end{pmatrix} = \frac{1}{-1} \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$$

So this matrix is its own inverse! You can double-check this by computing $M^2 = Id$.