18.02 Recitation
Solutions
28 September 2011

The problems with a (∗) be the number ask you to consult the topographic map on the back of the sheet. Let $h(x, y)$ denote the the altitude at a point with coordinates (x, y), where the origin is in Ingleby and a unit in the x or y direction is one foot.

1*. *How many peaks can you see along the ridge of Sawmill Mountain? How can you see the maximum of a function by looking at its level curves?*

   There are six peaks (including Sawmill Knob). These are the spots where the level curves close up into loops, with no other level curves inside.

2. *Describe the level curves of the function $g(x, y) = 4x^2 + y^2$.*

   First try $c = 4$. The level curve is the set of solutions to $4x^2 + y^2 = 4$. Now, if this were $x^2 + y^2 = 4$, it would be a circle of radius two. Instead, it's $(2x)^2 + y^2 = 4$, so it's squished by a factor of two in the x direction. The level curve is an ellipse. For other values of c we get other ellipses, scaled up by appropriate factors.

4. *Let $f(x, y) = (x + y)^2 + xy^2 + 2$. Compute the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.*

$$\frac{\partial f}{\partial x} = 2(x + y) + y^2, \qquad \frac{\partial f}{\partial y} = 2(x + y) + 2xy.$$

5*. *Suppose you stand at the point A on the map. What is the meaning of the partial derivative $\frac{\partial h}{\partial x}$? $\frac{\partial h}{\partial y}$? How could you estimate the values of these derivatives?*

   The derivative $\frac{\partial f}{\partial x}$ is the roughly number of feet our elevation increases when we walk one foot in the x direction, i.e. east. The derivative $\frac{\partial f}{\partial y}$ is the number of feet our elevation increases when we walk one foot in the y direction, i.e. north.

6. *Let f be the function from problem 4. Approximate $f(2+\Delta x, 1+\Delta y)$ using the partial derivatives you computed.*

   The formula for an approximation is

$$f(x + \Delta x, y + \Delta y) \approx f(x, y) + \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y.$$

   Using the partial derivatives computed in the earlier problem, and plugging in $(2, 1)$, this is

$$f(2 + \Delta x, 1 + \Delta y) \approx 13 + 7\Delta x + 10\Delta y.$$

   Thus $f(2.1, 0.9) \approx 13 + 7(0.1) + 10(-0.1) = 12.7$.

   The precise value is 12.701, so the approximation is quite good in this case.

7. *Find the tangent plane to the graph of the function f at $(2, 1, 13)$.*

We use the formula

$$z = f(a, b) + \frac{\partial f}{\partial x}(x - a) + \frac{\partial f}{\partial y}(y - b).$$

that I derived in recitation. This gives the values

$$z = 13 + 7(x - 2) + 10(y - 1)$$

8*. *Suppose that you want to run a railroad line roughly following the creek to Ingleby. How would you route it? The path should be as short as possible, but is limited to a 2% grade (i.e. slope at most* 0.02*). What does this have to do with the level curves?*

You can see the actual railroad route on the map – it's the path marked "abandoned". You want the track to change altitude as little as possible, which means that it should be more or less parallel to the level curves (and indeed, this is the case with the line that's actually there).