18.02 Recitation
Problems
03 October 2011

1. *(13.5.12) Find the critical points of* $z = 2xy\exp\left(-\frac{1}{8}(4x^2+y^2)\right)$.

   The derivatives are

$$f_x = 2xy(\exp\left(-\frac{1}{8}(4x^2+y^2)\right)(-x) + 2y\exp\left(-\frac{1}{8}(4x^2+y^2)\right)$$
$$= 2y(1-x^2)\exp\left(-\frac{1}{8}(4x^2+y^2)\right)$$
$$f_y = 2xy\left(\exp\left(-\frac{1}{8}(4x^2+y^2)\right)(-y/4)\right) + 2x\exp\left(-\frac{1}{8}(4x^2+y^2)\right)$$
$$= \frac{x}{2}(4-y^2)\exp\left(-\frac{1}{8}(4x^2+y^2)\right)$$

   These vanish when $y(1-x)(1+x) = 0$ and $x(2-y)(2+y) = 0$, namely the points $(0,0)$, $(1,2)$, $(1,-2)$, $(-1,2)$, $(-1,-2)$. You can classify the critical points using the second derivative test if you want.

2. *Consider the function* ax^2+by^2. *It has a critical point at* $(0,0)$. *When is it a minimum? When a maximum? When a saddle point? Sketch a graph of this function.*

   The Hessian is

$$\begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} 2a & 0 \\ 0 & 2b \end{pmatrix}$$

   The determinant is $H = 4ab$. The output of the second derivative test is the following:

a	b	H	Critical point
>0	>0	>0	minimum
<0	>0	<0	saddle
>0	<0	<0	saddle
<0	<0	>0	maximum

3. *(2F-1b). Find the point on the surface* $x^2-yz = 1$ *which is closest to the origin. (Hint: minimize the square of the distance, instead of the distance itself).*

   The square of the distance is $x^2+y^2+z^2$. Since $x^2 = 1+yz$ on the surface, we want to minimize $1+yz+y^2+z^2$ (a function of the independent variables y and z). So we'll use the second-derivative test:

$$\frac{\partial f}{\partial y} = 2y+z, \qquad \frac{\partial f}{\partial z} = y+2z.$$

   These vanish simultaneously at $(0,0)$, where $x = \pm 1$. So the two critical points of the distance are $(1,0,0)$ and $(-1,0,0)$. The second derivatives are

$$\begin{pmatrix} f_{yy} & f_{yz} \\ f_{zy} & f_{zz} \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

   The determinant is the Hessian is $+3$, so this is a minimum as expected.

4. *Find the maximum value of the function* $f(x, y) = (x - 1)^2 + y^2$ *on or inside a circle of radius* 2 *centered at* $(0, 0)$.

   First we find the critical points on the interior. $f_x = 2(x - 1)$ and $f_y = 2y$, so the unique critical point is $(1, 0)$. By inspection it is a minimum, and so the maximum of the circle is going to be on the boundary.

   To find the maximum value on the boundary, write $\gamma(t) = (2\cos t, 2\sin t)$. This parametrizes the circle, and so we just need to find the t for which $f(\gamma_x(t), \gamma_y(t))$ is as large as possible. Plugging in, we want to maximize

$$(2\cos t - 1)^2 + (2\sin t)^2 = 4\cos^2 t - 4\cos t + 1 + 4\sin^2 t = 5 - 4\cos t.$$

   This has a maximum at $t = \pi$, when $\cos t = -1$. The corresponding point is $\gamma(\pi) = (-2, 0)$ and the value of f there is 3.

5. *What is the maximum value of the area of a rectangle whose total perimeter is less than or equal to ten? (The answer may be obvious to you, but try to do the problem using our new methods)*

   Suppose the sides are x and y. We want to find the maximum value of the function $f(x, y) = xy$ on the region $x \geq 0$, $y \geq 0$, $2x + 2y \leq 10$. Note that the region is a triangle with vertices at $(0, 0)$, $(5, 0)$, and $(0, 5)$.

   First use the partial derivative test to find the largest value in the interior of the region:

$$\frac{\partial f}{\partial x} = y, \qquad \frac{\partial f}{\partial y} = x.$$

.

   So the only critical point is at $(0, 0)$, where the area is 0. It's pretty clear that this is not going to be a maximum. The other possibility to find a maximum is on the boundary. We'll find the maximum on each of the three edges.

   Among points $(x, 0)$, the area is always zero. Same goes for $(0, y)$. So look at the third side (this corresponds to rectangles with perimeter 10, which is of course where we're going to find the maximum. Here $y = 5 - x$, and we want to find the value of x maximizing $x(5 - x)$. This is just a single-variable problem. Take the derivative, set it to 0, and you get $x = 5/2$. So the maximum value of the function is at $(5/2, 5/2)$, where the area is $25/4$. This is a square.

6. There is a unique quadratic $y = 2x^2 + bx + c$ passing through the two points $(0, -4)$ and $(1, 1)$. Find b and c in two different ways:

   (a) First use the method that we used to find a regression line: consider the function of (a, b, c) whose value is the sum of the three errors. Set the three partial derivatives equal to 0 and solve for the coefficients.

   (b) Set up a linear system in three variables to find the right values of the coefficients.

   What if I had given you three points (throw in $(-1, -5)$) and not specified the leading coefficient as 2? Can you come up with a method to find all three coefficients?

7. *How would you do a least-squares regression to fit a quadratic* $y = x^2 + bx + c$ *to a set of points (assume the leading coefficient is* 1*). Check this in the case that only two points are given:* $(0, -4)$ *and* $(1, 1)$*.*
   What if the leading coefficient were not assumed to be 1*? Can you fit a quadratic of the form* $ax^2 + bx + c$*?*

   Least-squares regression is really just a special case of a minimization problem. In this case, the thing we want to minimize is

$$\text{SquaredError}(a, b) = \sum_{i=1}^{n} ((x_i^2 + ax_i + b) - y_i)^2.$$

   This would be 0 if we could find a and b such that $x_i^2 + ax_i + b = y_i$ for every i. Otherwise, the a and b minimizing it are still the best possible approximation.

   The derivatives are

$$\frac{\partial SE}{\partial a} = \sum_{i=1}^{n} 2x_i((x_i^2 + ax_i + b) - y_i)$$
$$\frac{\partial SE}{\partial b} = \sum_{i=1}^{n} 2((x_i^2 + ax_i + b) - y_i)$$

   Now set both of these equal to 0 and solve for a and b.

   To do a regression for three variables, we'd use a function of three variables:

$$\text{SquaredError}(a, b, c) = \sum_{i=1}^{n} ((ax_i^2 + bx_i + c) - y_i)^2.$$

   To minimize such a thing, you do what you might expect based on the two-variable case: set all three partial derivatives to 0 and solve for a critical point!