18.02 Recitation
Problems
19 October 2011

1. *Integrate the function $f(x, y) = 1$ over the unit circle. Does your answer make sense? Can you do the integral in polar coordinates?*

   This is easiest in polar coordinates. We want $0 \le \theta \le 2\pi$, $r \le 0 \le 1$, and we're just integrating $r\,dr\,d\theta$.

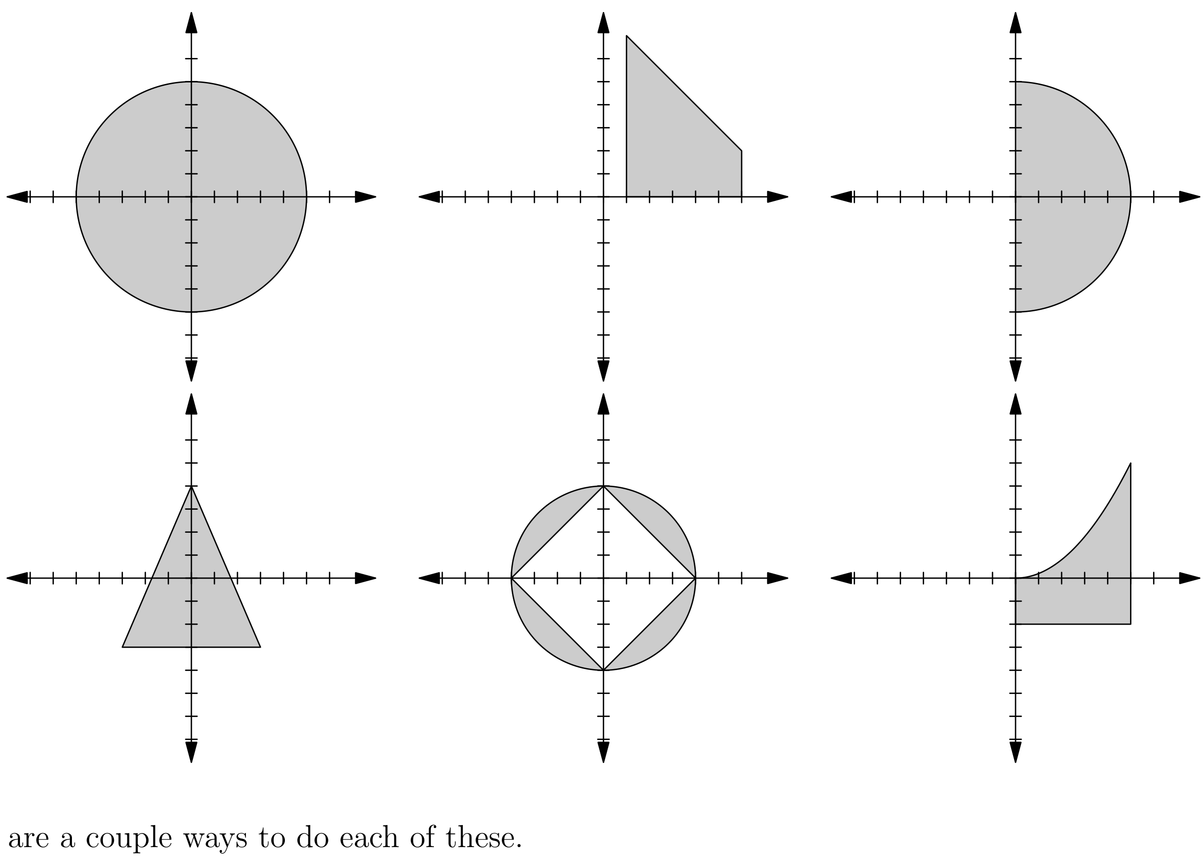
   $$\int_{r=0}^{1}\int_{\theta=0}^{2\pi} r\,d\theta\,dr = \int_{r=0}^{1} 2\pi r\,dr = (\pi r^2)\big|_0^1 = \pi,$$

   which is the area of the unit circle, as it should be.

   In rectangular coordinates we arrive at this as

   $$\begin{aligned}\int_{x=-1}^{1}\int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 1\,dy\,dx &= \int_{-1}^{1} 2\sqrt{1-x^2}\,dx \\ &= \left(x\sqrt{1-x^2} + \sin^{-1}(x)\right)\Big|_{-1}^{1} = \pi.\end{aligned}$$

2. *Set up limits of integration for integrals of $f(x, y)$ over the following regions. Can you find more than one way? In particular, try to use polar coordinates for the first, third, and fifth examples.*



   There are a couple ways to do each of these.

(a)

$$\int_{x=-1}^{1}\int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f(x,y)\,dy\,dx$$

$$= \int_{y=-1}^{1}\int_{x=-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x,y)\,dx\,dy$$

$$= \int_{r=0}^{1}\int_{\theta=0}^{2\pi} f(r,\theta)\,r\,d\theta\,dr.$$

(b)

$$\int_{x=1}^{6}\int_{y=0}^{8-x} f(x,y)\,dy\,dx$$

$$= \int_{y=0}^{2}\int_{x=1}^{6} f(x,y)\,dx\,dy + \int_{y=2}^{7}\int_{x=1}^{8-y} f(x,y)\,dx\,dy$$

(c)

$$\int_{x=0}^{1}\int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f(x,y)\,dy\,dx$$

$$= \int_{y=-1}^{1}\int_{x=0}^{\sqrt{1-y^2}} f(x,y)\,dx\,dy$$

$$= \int_{r=0}^{1}\int_{\theta=-\pi/2}^{\pi/2} f(r,\theta)\,r\,d\theta\,dr.$$

(d)

$$\int_{x=-3}^{0}\int_{y=-3}^{7/3x+4} f(x,y)\,dy\,dx + \int_{x=0}^{3}\int_{y=-3}^{-7/3x+4} f(x,y)\,dy\,dx$$

$$= \int_{y=-3}^{4}\int_{x=(-3/7)(y-4)}^{x=(3/7)(y-4)} f(x,y)\,dx\,dy$$

(e) Let's just do the top right piece of this region. The others are similar, and the total integral is obtained by adding them.

$$\int_{x=0}^{4}\int_{y=4-x}^{\sqrt{16-x^2}} f(x,y)\,dy\,dx$$

$$= \int_{y=0}^{4}\int_{x=4-y}^{\sqrt{16-y^2}} f(x,y)\,dx\,dy$$

$$= \int_{\theta=0}^{\pi/2}\int_{r=4/(\cos\theta+\sin\theta)}^{4} f(r,\theta)\,r\,dr\,d\theta.$$

Why the bounds on the last r? Well, for a fixed θ we want r to range from the lowest value where it meets the shaded region, to the outermost point of the region, which is at 4. The innermost point at angle θ has $y = 4 - x$, and our point has coordinates $(r\cos\theta, r\sin\theta)$. So we want $r\cos\theta = 4 - r\sin\theta$, whence $r = 4/(\cos\theta + \sin\theta)$.

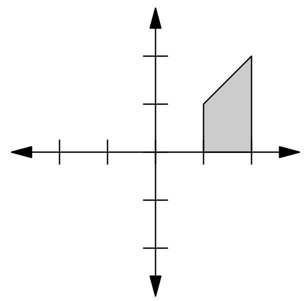
(f)

3. *Evaluate the integral*

$$\int_{x=1}^{2}\int_{y=0}^{x}\frac{1}{(x^2+y^2)^{3/2}}\,dy\,dx$$

by converting to polar coordinates.

The hard part is find the bounds of integration. Drawing the region,

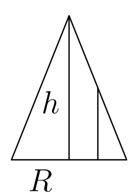


We'll integrate for $0 \le \theta \le \pi/4$. For a given value of θ, we need to find the bounds for r. A little trig shows we want $\sec\theta \le r \le 2\sec\theta$. Rewriting the function in terms of r and θ, we obtain

$$\begin{aligned}\int_0^{\pi/4}\int_{\sec\theta}^{2\sec\theta}\frac{1}{r^3}r\,dr\,d\theta &= \int_0^{\pi/4}\int_{\sec\theta}^{2\sec\theta}\frac{1}{r^2}\,dr\,d\theta\\ &= \int_0^{\pi/4}\left(-\frac{1}{r}\right)\Big|_{\sec\theta}^{2\sec\theta}\,d\theta = \int_0^{\pi/4}\left(\cos\theta-\frac{1}{2}\cos\theta\right)\,d\theta = \frac{\sqrt{2}}{4}.\end{aligned}$$

4. *Find the volume of a circular cone with base a circle of radius R and vertex at $(0,0,H)$.*

We'll do this in polar coordinates. Think about a cross-section of the cone.



The height over (r,θ) depends only on r. We can see similar triangles: $(R-r)/h(r,\theta) = R/H$. So $h(r,\theta) = H/R\cdot(R-r)$. Then

$$\begin{aligned}V &= \int_{r=0}^{R}\int_{\theta=0}^{2\pi}\frac{H(R-r)}{R}r\,d\theta\,dr = \int_{r=0}^{R}\frac{2\pi H}{R}r(R-r)\,d\theta\,dr\\ &= \frac{2\pi H}{R}\left(\frac{Rr^2}{2}-\frac{r^3}{3}\right)_{r=0}^{R} = \frac{\pi R^2 H}{3},\end{aligned}$$

which agrees with the usual formula for the volume of a cone.