18.02 Recitation
Problems
26 October 2011

1. *Find the area of a parallelogram with sides $y - x = 0$, $y - x = 2$, $3x - y = 0$, $3x - y = 4$.*

   To find the area of anything, we need to integrate the function $f(x, y) = 1$ over it. In order to make this easier, we'll use the change of coordinates $u = y - x$, $v = 3x - y$. In these coordinates, the sides of the parallelogram are just $v = 0$, $v = 4$, $u = 0$, and $u = 2$. Solving the linear equations, this gives us $x = (u + v)/2$ and $y = (3u + v)/2$, so the Jacobian is $|\begin{vmatrix} 1/2 & 1/2 \\ 3/2 & 1/2 \end{vmatrix}| = |1/4 - 3/4| = 1/2$.

   Now we integrate to find the area:

$$A = \int_{v=0}^{4} \int_{u=0}^{2} 1 \left(\frac{1}{2}\, du\, dv \right) = 4.$$

2. *Find the area of the region defined by $x^{2/3} + y^{2/3} = 1$ in the first quadrant.*

   Again we integrate the function 1. Take $u = x^{2/3}$, $v = y^{2/3}$. Our region is bounded by the three curves $x = 0$, $y = 0$, $x^{2/3} + y^{2/3} = 1$. In terms of u and v, these are respectively $u = 0$, $v = 0$, $u + v = 1$, which is just a triangle. We have $x = u^{3/2}$ and $y = v^{3/2}$, and so the Jacobian is $J = \begin{vmatrix} 3\sqrt{u}/2 & 0 \\ 0 & 3\sqrt{v}/2 \end{vmatrix}$. So the integral is

$$\int_{u=0}^{1} \int_{v=0}^{1-u} \frac{9}{4} \sqrt{uv}\, dv\, du = \frac{9}{4} \frac{\pi}{24} = \frac{3\pi}{32}.$$

   It's a bit of work to actually do the integral, but I omit the calculation.

3. *(14.9.12) Integrate $f(x, y) = \frac{1}{(x^2+y^2)^2}$, over the region in the first quadrant bounded by the circles $x^2+y^2 = 2x$, $x^2+y^2 = 6x$, and $x^2+y^2 = 2y$, $x^2+y^2 = 8y$. Use $u = 2x/(x^2+y^2)$ and $v = 2y/(x^2+y^2)$.*

   First let's work out the limits of integration in the new coordinates. The circle $x^2+y^2 = 2y$ translates to $\frac{2y}{x^2+y^2} = 1$, which is just the condition $v = 1$. In the same way, the other three boundaries are expression in terms of u and v as $u = 1$, $u = 1/3$, $v = 1/4$. So we're just going to be integrating over a rectangle. The hard part is to compute the Jacobian and expression the function in our new coordinates. Observe that

$$u^2 + v^2 = \frac{4x^2}{(x^2 + y^2)^2} + \frac{4y^2}{(x^2 + y^2)^2} = \frac{4}{x^2 + y^2}.$$

   This looks quite similar to our function, and it's easy to see that f should be written in terms of u and v as $((u^2 + v^2)/4)^2 = (u^2 + v^2)/16$. We also get $x = 2u/(u^2 + v^2)$ and $y = 2v/(u^2 + v^2)$. This allows us to compute the Jacobian as

$$\begin{aligned} J &= |\begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}| \\ &= |\begin{vmatrix} \frac{2(u^2+v^2)-2u(2u)}{(u^2+v^2)^2} & -\frac{4uv}{(u^2+v^2)^2} \\ -\frac{4uv}{u^2+v^2} & \frac{2(u^2+v^2)-2v(2v)}{(u^2+v^2)^2} \end{vmatrix}| \\ &= \cdots = \frac{4}{(u^2 + v^2)^2}. \end{aligned}$$

   So the integral is

$$\int_{u=1/3}^{1} \int_{v=1/4}^{1} \frac{(u^2 + v^2)^2}{16} \frac{4}{(u^2 + v^2)^2}\, dv\, du = \left(\frac{1}{4}\right) \left(\frac{2}{3}\right) \left(\frac{3}{4}\right) = \frac{1}{8}.$$

4. *Find the center of mass of a right half-disk of radius a centered at* $(0, 0)$*, using the coordinate system of your choice.*

We'll do this in polar coordinates. The area of the region is $\pi/2$ since it's a half-circle. The y coordinate of the center of mass is obviously 0 by symmetricy. The x coordinate is the intgegral of the function x with respect to area, which in polar coordinates is expressed as $r \cos\theta$. Thus our answer is

$$\bar{x} = \frac{2}{\pi} \int_{r=0}^{1} \int_{\theta=-\pi/2}^{\pi/2} r \cos\theta r \, dr \, d\theta = \frac{2}{\pi} \int_{0}^{1} 2r^2 \, dr = \frac{2}{\pi}\frac{2}{3} = \frac{4}{3\pi}.$$

So the center of mass is $(4/(3\pi), 0)$.