18.02 Recitation
Problems
9 November 2011

1. *Consider the field $\vec{F} = \hat{\imath} + \hat{\jmath}$. How should a line segment be oriented so that the flux of $\vec{F}$ across the segment is maximized? Minimized? Zero?*

   It's maximized when the segment is perpendicular to the flux, so that the normal is parallel to the field $\vec{F}$. It's minimized when the line segment is perpendicular, but pointing in the other direction (so the normal is opposite the field). It's zero when the segment is parallel to the field.

2. *(4F3) Verify Green's theorem in the normal form by calculating both sides and showing they are equal if $\vec{F} = x\,\hat{\imath} + y\,\hat{\jmath}$, and C is formed by the upper half of the unit circle and the x-axis interval* $[1, 1]$.

   This one's written up in the course notes.

3. *Compute the area of a triangle with vertices at* $(0, 0)$, $(1, 0)$, *and* $(0, 1)$ *in the hardest possible way: use Green's theorem to convert the double integral for area into a line integral and evaluate.*

   We'll use the formula $\iint_R 1\,dA = \oint_C x\,dy$, which you checked on the homework. Here C is a loop that follows the edges of the triangle, and to integrate over it, we need to split it into three piece. These are parametrized by $\vec{r}_1(t) = \langle 1 - t, t\rangle$, $\vec{r}_2(t) = \langle 0, 1 - t\rangle$, $\vec{r}_3(t) = \langle t, 0\rangle$.

   So
   $$A = \int_{C_1} 1 - t\,dt + \int_{C_2} 0\,(-dt) + \int_{C_3} t\,(0\,dt) = \frac{1}{2} + 0 + 0 = \frac{1}{2},$$
   which is the right answer.

4. *Which of these regions are simply connected?*

   (a) *A star-shaped set* This is simply connected. The set isn't convex, but that doesn't change the fact that it has no holes.

   (b) *The unit circle* Not simply connected.

   (c) *The unit disk* Simply connected.

   (d) *The right half-plane* Simply connected.

   (e) *An annulus* Not simply connected.

5. *We've seen that*
   $$\int_C -\frac{y}{x^2 + y^2}\,dx + \frac{x}{x^2 + y^2}\,dy = 2\pi$$
   where C is a circle around the origin. What is the integral over a path that goes twice around the origin?

Any time we go around path twice, the integral just doubles! So in this case it's 4π. A path that goes around the origin n times will have integral $2\pi n$. n here is called the winding number. The interesting thing is that you can tell how many times a path loops around the origin by computing a certain line integral.

6. *Think about the integral*

$$\int_C \left(-\frac{y}{x^2+y^2} - \sqrt{2}\frac{y}{(x-2)^2+y^2}\right) dx + \left(\frac{x}{x^2+y^2} + \sqrt{2}\frac{x-2}{(x-2)^2+y^2}\right) dy.$$

What is its integral around a circle of radius 1 *centered at* $(0,0)$*? At* $(2,0)$*? Describe the integral around some other paths of your choosing.*

This is a fun one, but maybe a bit beyond what we need to worry about. If you want to discuss it stop by hours office sometime.

7. *Set up the triple integral in rectangular coordinates to compute the center of mass of a hemisphere. What are the bounds? If you can, evaluate the integral (this will be hard – later we will study spherical coordinates, in which it's more tractable)!*

Let's assume for simplicity that we're looking at the hemisphere defined by $x^2+y^2+z^2 \leq 1$, $z \geq 0$. The whole thing takes place over the unit circle, so we want x from -1 to 1 and y from $-\sqrt{1-x^2}$ to $\sqrt{1-x^2}$. For fixed x and y, we want $x^2+y^2+z^2 = 1$, so z should range from 0 to $\sqrt{1-x^2-y^2}$. The center of mass clearly has x and y coordinate equal to 0, and the z part is computed as

$$\int_{x=-1}^{1} \int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{z=0}^{\sqrt{1-x^2-y^2}} z\,dz\,dy\,dx$$