1. Double integrals (3A-2, 3A-3, 3D-5)

   (a) $\iint_R f(x,y)\,dA$ gives the volume under the graph of $f(x,y)$ in the region R.

   (b) Three steps to compute it: find the bounds of integration, evaluate the inner integral, evaluate the outer integral.

   (c) For a rectangle, bounds are:

   $$\int_{x=a}^{b}\int_{y=c}^{d} f(x,y)\,dy\,dx$$

   (d) In general the bounds on the inner integral will depend on the outer variable. If x is between 1 and 2, and y is between x and x^2, the bounds are $\int_{x=1}^{2}\int_{y=x}^{x^2} f(x,y)\,dy\,dx$. (see notes 16 for some examples of setting this up)

   (e) The variable in the inner integral should have a range that depends on the outer variable (not the other way around!)

   (f) In evaluating the inner integral, treat the variable of the outer integral (x, in the above examples) as a constant, and integrate as usual. The result should be a function of x.

   (g) Then do the outer integral and you're done!

   Here are some particular double integrals worth keeping in mind:

Property	Integral
Area	$\iint_R 1\,dx\,dy$
Average value of function	(1/area) $\iint_R f(x,y)\,dx\,dy$
Center of mass: x coordinate	$1/M\ \iint_R x\delta(x,y)\,dx\,dy$
Center of mass: y coordinate	$1/M\ \iint_R y\delta(x,y)\,dx\,dy$
Moment of inertia about x-axis	$\iint_R \delta(x,y)y^2\,dx\,dy$
Moment of inertia about y-axis	$\iint_R \delta(x,y)x^2\,dx\,dy$
Polar moment of inertia	$\iint_R \delta(x,y)(x^2+y^2)\,dx\,dy$

2. Change of variables (3A-5, 3B-2, 3D-1, 3D-5)

   (a) Often an integral will be made easier by switching the order of integration. To do this, we need to reexpress the bounds on the same region but with the other order for the variables. Sketch the region, using the given bounds, and then figure out the bounds with the new order of integration.

   (b) You can also compute double integrals in polar coordinates. It works the same way: if the bounds on r depend on θ (the typical situation), then you should have dr as the inside integral, $d\theta$ as the outside one. An important thing to keep in mind is that the area element dA is expressed in polar as $r\,dr\,d\theta$. Don't forget the leading factor of r!

   (c) It's also useful to be able to change the coordinate system of an integral to new coordinates $u(x,y)$, $v(x,y)$. Usually a good first step is to figure out how to write x and y in terms of u and v. This will make the other steps easier.

   To do this, there are three basic steps:

   i. Express the bounds on the integral in terms of u and v. To do this, take the equation defining each edge of the region as an equality in terms of u and v.

   ii. Express the function $f(x,y)$ as a function of u and v.

   iii. Express the area element in terms of du and dv. This is done by writing

   $$du\,dv = |\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}|\,dx\,dy.$$

   The matrix $\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$ is called the *Jacobian*. Don't forget to take the absolute value!

3. Vector fields (4A-1, 4A-3)

   (a) A vector field assigns to each point (x,y) a vector $\langle M(x,y), N(x,y)\rangle$.

   (b) Two important examples: $\vec{F}_1(x,y) = \langle x,y\rangle$. The vector at (x,y) points directly away from the origin. $\vec{F}_2(x,y) = \langle -y,x\rangle$ gives a field tangent to circles centered at the origin. Be sure to remember these!

(c) One way to get a vector field is to start with a function $f(x,y)$, and take $\vec{F}(x,y)$ to be the gradient of the function at each point. A vector field that arises in this way is called a *gradient field*.

(d) We can associate to a vector field two different functions, the divergence and the curl. These are defined by

$$\operatorname{curl}\vec{F} = N_x - M_y, \qquad \operatorname{div}\vec{F} = M_x + N_y.$$

(e) If a field is a gradient field, it's of the form $\langle \partial f/\partial x, \partial f/\partial y\rangle$. This implies that it has curl equal to 0.

(f) The converse is almost true as well: if a function is defined on a simply connected region, and it has curl equal to 0, then it's a gradient field.

(g) Be sure you know how to find the function of which it's a gradient (the potential function)! Two ways to do this are explained in Lecture Notes 21.

(h) "Simply connected" just means the region has no holes in it. Examples are the whole plane, half plane, and unit disk. Non-examples are a plane with a hole in it, an annulus, or a plane with a line segment removed.

(i) If the region isn't simply connected, a field can have curl zero but fail to be conservative. Be on the lookout whenever you see a field that looks like $\vec{F} = \left\langle -y/(x^2+y^2), x/(x^2+y^2)\right\rangle$.

4. Line integrals (4B-1, 4B-2, 4C-3, 4C-5)

(a) Given a vector field $\vec{F}(x,y) = M(x,y)\hat{\imath} + N(x,y)\hat{\jmath}$, define the integral along a curve C parametrized by $\vec{r}(t) = \langle x(t), y(t)\rangle$,

$$\int_C \vec{F}\cdot d\vec{r} = \int_{t=a}^{b} \left\langle \vec{F}(x(t), y(t))\right\rangle \cdot \langle x'(t), y'(t)\rangle \, dt.$$

(b) You'll also see this written out with $dx = x'(t)\,dt$, as

$$\int_C \vec{F}\cdot d\vec{r} = \int_C M(x(t),y(t))\,dx + N(x(t),y(t))\,dy.$$

(c) Either way, what you need to do is parametrize the curve, and plug in the $x(t)$ and $y(t)$ you use into the expression for the integral.

(d) The fundamental theorem of calculus for line integrals states that if the vector field we're integrating is a gradient field $\vec{F} = \nabla f$, we can save a lot of trouble just by evaluating the function f at the endpoints of the path

$$\int_C \vec{F}\cdot d\vec{r} = f(b) - f(a),$$

where b is the endpoint of C, and a its starting point. In particular this means that the value of the integral doesn't depend on the path taken, so gradient fields are also called conservative fields.

5. Green's theorem (4D-1, 4D-5, 4D-7, 4E-3)

(a) Green's theorem lets us convert the line integral of a vector field around a closed loop to the integral of a related function over the inside of the region.

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_R \operatorname{curl}\vec{F}\,dA.$$

(b) Writing this out explicitly in terms of differentials,

$$\oint_C M\,dx + N\,dy = \iint_R N_x - M_y\,dy\,dx.$$

(c) We can also consider the flux $\vec{F}\cdot\hat{n}$. This is a measure of how much the vector field "flows" across the curve C. There's another form of Green's theorem for flux, which says

$$\oint_C \vec{F}\cdot\hat{n}\,ds = \iint_R \operatorname{div}\vec{F}\,dA.$$

(d) As differentials,

$$\oint_C M\,dy - N\,dx = \iint_R M_x + N_y\,dy\,dx.$$