1 Inverting a 3×3 matrix

We saw in recitation that it's easy to invert a 2×2 matrix. Just use the formula:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \Longrightarrow \quad A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

If A is 3×3, there isn't such a simple formula. We have to do something more complicated. I'll write it up here, but if you're still confused you can also try looking in section M of the supplemental notes. This contains all the material about matrices that you need to worry about.

First we need a couple definitions. Given a 3×3 matrix, the ij-minor is the determinant of the matrix that's left over after you delete the i^{th} row and the j^{th} column. Let's use the matrix

$$A = \begin{pmatrix} 3 & 1 & 8 \\ 2 & 5 & -4 \\ 1 & -2 & -3 \end{pmatrix}$$

from the problem sheet. To find the 12-minor, we delete the first row and second column, obtaining a 2×2 matrix.

$$\begin{pmatrix} X & X & X \\ 2 & X & -4 \\ 1 & X & -3 \end{pmatrix} \quad \Longrightarrow \quad \begin{pmatrix} 2 & -4 \\ 1 & -3 \end{pmatrix}$$

The minor is the determinant of this 2×2 matrix. In this case, that's $(2)(-3) - (1)(-4) = -2$. We refer to the ij-minor using the notation $|A_{ij}|$.

We're going to need to compute all the minors of A to find its inverse, so let's get this out of the way now:

$$|A_{11}| = \begin{vmatrix} 5 & -4 \\ -2 & -3 \end{vmatrix} = -23 \qquad |A_{12}| = \begin{vmatrix} 2 & -4 \\ 1 & -3 \end{vmatrix} = -2 \qquad |A_{13}| = \begin{vmatrix} 2 & 5 \\ 1 & -2 \end{vmatrix} = -9$$

$$|A_{21}| = \begin{vmatrix} 1 & 8 \\ -2 & -3 \end{vmatrix} = 13 \qquad |A_{22}| = \begin{vmatrix} 3 & 8 \\ 1 & -3 \end{vmatrix} = -17 \qquad |A_{23}| = \begin{vmatrix} 3 & 1 \\ 1 & -2 \end{vmatrix} = -7$$

$$|A_{31}| = \begin{vmatrix} 1 & 8 \\ 5 & -4 \end{vmatrix} = -44 \qquad |A_{32}| = \begin{vmatrix} 3 & 8 \\ 2 & -4 \end{vmatrix} = -28 \qquad |A_{33}| = \begin{vmatrix} 3 & 1 \\ 2 & 5 \end{vmatrix} = 13$$

Now form the matrix of *cofactors*. The ij-cofactor is just the $(-1)^{i+j}\,|A_{ij}|$. Make a matrix by sticking all the cofactors into a 3×3 matrix. This is the same as just building a matrix out of all the minors we just computed, but introducing signs in the following pattern:

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

In general, the matrix is

$$\begin{pmatrix} |A_{11}| & -|A_{12}| & |A_{13}| \\ -|A_{21}| & |A_{22}| & -|A_{23}| \\ |A_{31}| & -|A_{32}| & |A_{33}| \end{pmatrix}$$

In our case the matrix of cofactors is

$$\begin{pmatrix} -23 & 2 & -9 \\ -13 & -17 & 7 \\ -44 & 28 & 13 \end{pmatrix}$$

This matrix is called the (classical) adjoint, or the adjugate. We're almost done. Now you just need to take the transpose (i.e. flip the matrix over its diagonal), and multiply by $1/\det A$:

$$A^{-1} = \frac{1}{|A|} \begin{pmatrix} |A_{11}| & -|A_{12}| & |A_{13}| \\ -|A_{21}| & |A_{22}| & -|A_{23}| \\ |A_{31}| & -|A_{32}| & |A_{33}| \end{pmatrix}^T$$

In the example, we computed in section that the determinant is -139. So

$$A^{-1} = -\frac{1}{139} \begin{pmatrix} -23 & -13 & -44 \\ 2 & -17 & 28 \\ -9 & 7 & 13 \end{pmatrix}$$

This whole procedure is rather tedious and it's easy to make a mistake. To check your answer, you can just compute the matrix product $A^{-1}A$. If you did everything right, you'll get the 3×3 identity matrix

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Fortunately there is a slicker way to compute matrix inverses, but it requires making a few more definitions. You'll do lots of this if you ever take 18.06.