Today in recitation we talked about the fact that the function

$$f(t) = \sin t + \sin(\pi t)$$

isn't periodic at all, because one of the summands has period 2π, and the other summand has period 2. Since π is an irrational number, there is no number that is a period of both. If you weren't in class today, you can look at a plot on Wolfram Alpha: `http://bit.ly/GMOOLM`

On the other hand, I claimed that 710 is "close" to being a period, in the sense that

$$|f(t + 710) - f(t)| \leq 0.000061.$$

Why would this be?

On closer inspection, the explanation isn't so hard. Use the sum formula $\sin(x + y) = \sin x \cos y + \cos x \sin y$, and you find that

$$\begin{aligned} f(t + 710) - f(t) &= (\sin(t + 710) + \sin(\pi(t + 710))) - (\sin(t) + \sin(\pi t)) \\ &= \sin t \cos 710 - \sin t + \cos t \sin 710 - \sin(\pi t) + \sin(\pi t + 710\pi) \\ &= (1 - \cos 710) \sin t + \cos t \sin 710 - \sin(\pi t) + \sin(\pi t) \\ &= (1 - \cos 710) \sin t + \cos t \sin 710. \end{aligned}$$

Now, whip out a calculator and you find that

$$\begin{aligned} \cos 710 &= 0.999999998182636 \ldots \\ \sin 710 &= 0.000060288706691 \ldots \end{aligned}$$

The absolute value of $(1 - \cos 710) \sin t$ is less than $1 - \cos 710 \approx 1.8 \cdot 10^{-9}$ (since $\sin t$ is between -1 and 1). Similarly, $\cos t \sin 710$ has absolute value less than $\sin 710$, which is about $6.0 \cdot 10^{-5}$. So $|f(t + 710) - f(t)|$ is extremely small, less that 0.000061 as I claimed.

This isn't an entirely satisfactory answer – what's so special about 710 that made its cosine and sine so close to 1 and 0? This is the real mystery in the story. Well, since the cosine is about 1, it must be that 710 is very close to a multiple of 2π. And in fact, dividing 710 by 2π we discover that

$$113 \cdot 2\pi \approx 709.9999397113$$

This is equivalent to the fact that $\pi \approx 355/113$. That 44 is also "almost" a period boils down to the fact that $\pi \approx 22/7$ is also a pretty good approximation.

$$\pi = 3.141592654\ldots$$
$$\frac{355}{113} = 3.141592920\ldots$$
$$\frac{22}{7} = 3.142857143\ldots$$

The first approximation is good to six digits!

Here's a question related to the one we started with: the function $f(t)$ is certainly between -2 and 2. It'll never reach 2, since t and πt can't both be of the form $\pi/2 + 2\pi k$. But can it be greater than 1.9? 1.99? I won't say anything about this, but you can try to use rational approximations of π to show that it can indeed get arbitrarily close to 2. Life might be easier if you work with the function $\cos t + \cos(\pi t)$ instead.

Anyway, it looks like finding "near-periods" of $\sin t + \sin(\pi t)$ isn't so bad, we just need to find rational numbers that are close to π and multiply the numerator by 2. How hard can that be? What if we try $\pi \approx \frac{314}{100}$? This is good to two digits, so it ought to work as well as 22/7 did...

$$\cos 628 = 0.949697$$
$$\sin 628 = -0.313172$$

This is not even close! The cosine isn't *so* far from 1, but the sine isn't anywhere near 0 (this isn't surprising; the Taylor expansion for cosine has x^1 coefficient equal to 0, so it is more forgiving of deviations).

What went wrong? The issue is that $200\pi = 628 - 200\left(\frac{314}{100} - \pi\right)$. The difference inside the parentheses is fairly small, but it's multiplied by 200 before we take sine. In fact the error $314/100 - \pi$ is itself on the order of $\frac{1}{100}$ (since we just truncated the decimal expansion), so 614 should differ from 200π by something on the order of 1, and that makes a big difference when we take sine. We need to use an approximation such that the error is small even when multiplied by the denominator.

For example, we could hope to find p and q such that $\pi \approx p/q$, with error on the order of $1/q^2$, the reciprocal of the square of the denominator. Then even after we multiply by the denominator, the deviation between $q \cdot 2\pi$ and $2p$ still won't be very big (it'll be the size of $1/q$). So we aim to find to find integers p and q such that

$$\left|\pi - \frac{p}{q}\right| < \frac{1}{q^2}$$

It's a theorem proved by Dirichlet in the 1800s that given any irrational number, you can find infinitely many approximations by rational numbers that satisfy the above bound. For π, 22/7 and 355/113 are examples, but 314/100 is not. It's these especially good approximations that give things like $f(t + 710) \approx f(t)$.

There's a lot of very interesting math connected with these sorts of ideas – look up "Diophantine approximation" on Wikipedia if you're curious. For example, we might ask if it is always possible to find infinitely many approximations of an irrational number such that the error is the size of $1/q^3$. This is a tricky question, and the answer is generally negative: a deep result called Roth's theorem implies that for certain irrational numbers, $1/q^2$ error is the best you can manage.