This is only a brief summary of the topics so far – be sure to study the notes as well!

The main topic so far has been first order ODEs: $y' = f(x,y)$. We've developed techniques to qualitatively study their behavior by drawing a slope field and to numerically approximate a solution using Euler's method. We've also studied some particular classes of equations where additional methods are available.

General methods

(1.3.3, 1.3.6; sketch and identify fences, 6.14)

Given any old first order ODE, it's usually not possible give an equation for the solution(s). But it's still possible to say quite a bit about them, either by sketching an approximation of the graph, or approximating the values of a solution at many points.

The basic technique to deal with these things is drawing a direction field (also known as a slope field), which gives a visual representation of solutions. Things to know:

- Draw a *direction field* by sketching a few *isoclines*. Sketch integral curves after you've drawn the field.
- Be able to prove that a particular isocline is a *fence* (there are examples of this in the first homework and the second recitation notes). Identify a *funnel* and explain its implications for the asymptotics of solutions.
- A *separatrix* is a solution such that nearby solutions have dramatically different behavior.

There is one other tool we have that works for any first-order equation (rather than just those having some specific form): Euler's method. This gives a way to numerically approximate the solution to an equation. The basic mechanism is to start with a point (x_0, y_0). Increase x by the step size h should cause an increase in y of about $y'(x_0) \cdot h = f(x_0, y_0) \cdot h$. In general we move from (x_k, y_k) to (x_{k+1}, y_{k+1}) using the rule

$$x_{k+1} = x_k + h, \quad y_{k+1} = y_k + f(x_k, y_k) \cdot h.$$

There's an easy way to put all of this in a table, which is probably worth looking over.

Some classes of equations

(separable: 1.4.16, 1.4.34; autonomous: 1.7.5, 1.7.10)

The easiest case is when you have a *separable equation*: this is when $y' = f(x)g(y)$, and so you can write $dy/g(y) = f(x)\,dx$, integrate both sides, and then solve for y to get the general solution.

Another important class of functions is *autonomous equation*. This is when $f(x,y) = g(y)$ doesn't depend on x at all: this means that the change in y only depends on its current value. These are separable, but often the integrals involved in solving them are messy at best. On the other hand, it's easy to give simple qualitative descriptions of the solutions.

The *equilibria* are the values of y for which $g(y) = 0$. A solution will generally increase or decrease until it reaches one of the equilibria, or else increase or decrease without limit.

The way to see what happens is to draw the *phase line*. This shows in which ranges of y the solution will increase or decrease, and thus which equilibrium it will tend towards. When drawing the phase line, it can help to draw a plot of $g(y)$ against y; then you can see for which ranges of y the derivative is either positive or negative. It also lets you identity the stable, unstable, and semistable equilibria: know how to identify and interpret these!

We have two special classes of autonomous equations which model some kind of population growth.

- Natural growth: $y' = k_0 y$.
- Logistic growth: $y' = k_0(1 - (y/p))y$. The growth will slow as y approaches the *carrying capacity* p, since this means y' goes to 0. It is possible to give an exact solution, but often the qualitative behavior is all we care about.

If an equation involves a parameter, say a, the equilibria will depend on the value of a (you solve $y' = 0$ for y, and the answers will depend on a). A *bifurcation diagram* illustrates this dependence.

First order linear equations

(1.7.5, 1.7.10)

These are equations that only depend on x and x', and do so in a particularly simple way: they can be written in standard form $r(t)\,x'(t) + p(t)\,x(t) = q(t)$. Divide through by r to get *reduced standand linear form* $x'(t) + p(t)\,x(t) = q(t)$. This is a class of equations for which we can often find exact solutions.

Three important examples are computing compound interest with continuous depositing, simple RC circuits, and Newtonian cooling. You can read how to set these up in lecture notes 4 and 5.

There's a standard procedure for solving a first-order linear equation:

1. Solve the *associated homogeneous equation* $x'(t) + p(t)x(t) = 0$ (ignore q, basically). This is a separable equation: solve it, and you get $x_h = Ce^{-\int p(t)\,dt}$.

2. Once you solve the homogeneous equation you can modify it to find the solution to the inhomogeneous form.

3. Method one (variation of parameters): assume the general solution is ux_h, where x_h is some solution to the homogeneous equation. Plugging everything in gives $u = \int x_h(t)^{-1}q(t)\,dt$. Compute that integral, and then ux_h is your solution (details in notes #5).

4. Method two (integrating factor; you're going to get the same answer either way). Multiply through be $1/x_h$. Recognize the thing as the left as the derivative $\frac{d}{dt}(\text{something}(t) \cdot y)$, integrate both sides, and then solve for y.

Sometimes the language of systems and signals clarifies exactly what's going on in these situations. Think of the function $q(t)$ as an input, upon which the solution $y(t)$ somehow depends (it's the output). Initial conditions should be considered too. There's a sort of diagram you can draw – see lecture notes 4.

Complex numbers

Complex numbers have the form $a + bi$, where $i = \sqrt{-1}$. a is called the *real part*, b the *imaginary part*. You can multiply $(a + bi) \cdot (c + di)$ by using "foil" and remembering that $i^2 = -1$. The conjugate of $z = a + bi$ is the complex number $\bar{z} = a - bi$.

It's often useful to express complex numbers in polar coordinates. This works just like converting between polar and rectangular coordinates did in 18.02. r is called the modulus (or magnitute), and θ's called the argument). The key fact here is Euler's formula, which defines how to raise e to a complex power: $e^{i\theta} = \cos\theta + i\sin\theta$. Multiplying in polar coordinates is simple, as the moduli multiply and the arguments add: $(r_1e^{i\theta_1}) \cdot (r_2e^{i\theta_2}) = r_1r_2e^{i(\theta_1+\theta_2)}$. This observation also lets you find the n^{th} roots of complex numbers. You can use this to give slick proofs of pretty much every trig identity. There's an example on the second problem set.

Sinusoidal functions

These are functions of the form $f(t) = A\cos((2\pi/P)(t - t_0))$. A here is the amplitude, which measures how large the maxima are. P is the period with which it repeats (if $P = 2\pi$ this gives $f(t) = A\cos(t - t_0)$, as it should). t_0 is the time lag, the time until the first maximum. If $t_0 = 0$ there's a maximum at $t = 0$. Other constants which derive from these are the frequency $\nu = 1/P$, angular frequency $\omega = 2\pi/P$, and phase lag $\phi = \omega t_0$. In terms of these, we may write $f(t) = A\cos(\omega t - \phi)$, which is the "standard form".

Two important things to remember: if you add two sinusoidal functions with the same period, you get another one. The formulas are $A\cos(\omega t - \phi) = a\cos(\omega t) + b\sin(\omega t)$, where $A = \sqrt{a^2 + b^2}$, $a = A\cos\phi$, and $b = A\sin\phi$. Also know how to integrate things of the form $\int e^{rt}\cos(bt)\,dt$: the idea is that this is the real part of $e^{(r+bi)t}$, which is a much easier thing to integrate.