Exam I review solutions

If any of these solutions are indecipherable or outright wrong, let me know and I'll post a correction!

1. *(1.3.6) Consider the differential equation* $\frac{dy}{dx} = x - y + 1$. *Sketch some isoclines and describe the behavior of the integral curves. Which solutions have local maxima or minima? Points of inflection? Can you identify any fences?*

   The plot is in the book. Every solution seems to converge to the line $y = x$. Minima occur where $y' = 0$, which is the nullcline $y = x + 1$. Inflection points have $y'' = 0$, so $1 - y' = 0$. These occur on the $m = 1$ isocline, which is itself a solution. So no other solutions have inflection points.

   I claim that every isocline is eventually a fence, which explains this observation. For concreteness, let's consider the $m = 2$ isocline. It's defined by $2 = x - y + 1$, i.e. $y = x - 1$. Above the isocline, but below $y = x$, things have slope between 1 and 2. So they can't cross back below the isocline itself, since it has slope 1.

   This example doesn't really show the trickier things that can happen with fences. In general the thing to do is find the equation for the isocline and take its derivative; you'll get a fence when the slope of the m-isocline is always greater than (or always less) than m in some range. Check out the second recitation solutions for a harder example.

2. *(1.4.17) Consider the equation* $y' = 1 + x + y + xy$. *Find the general solution in two different ways: first by observing that it's separable, and then by putting it in reduced standard linear form and applying our general method.*

   Method I:

$$\begin{aligned} y' &= (1+x)(1+y) \\ \frac{dy}{1+y} &= (1+x)\,dx \\ \log(1+y) &= \left(\frac{x^2}{2} + x\right)\,dx \\ y &= = Ce^{x^2/2+x} - 1 \end{aligned}$$

   Method II:

   Reduced standard form is $y' - (1+x)y = 1 + x$. First solve the homogeneous:

$$\begin{aligned} \frac{dy}{y} &= (1+x)\,dx \\ \log y &= \frac{x^2}{2} + x + c \\ y &= Ce^{x^2/2+x} \end{aligned}$$

So $y_h = e^{x^2/2+x}$ is the solution to the associated homogeneous equation. First by variation of parameters:

$$u = \int \frac{1+x}{e^{x^2/2+x}}\,dt = -e^{-x^2/2-x} + c.$$

Then the general solution is

$$y = \left(-e^{-x^2/2-x} + c\right)\left(e^{x^2/2+x}\right) = Ce^{x^2/2+x} - 1,$$

which agrees with our first answer.

3. *(1.5.13) Solve $y' + y = e^x$, subject to $y(0) = 1$.*

   The solution to the homogeneous is e^{-x}. So $u = \int (e^x)(e^x)\,dx = \frac{1}{2}e^{2x} + c$ and

$$y = \frac{1}{2}e^x + ce^{-x}.$$

   To get $y(0) = 1$, we need $1 = \frac{1}{2} + c$, so $c = \frac{1}{2}$.

4. *Consider the logistic equation with stocking/harvesting*

$$\frac{dy}{dt} = y(2-y) + a.$$

   What are the equilibria, in terms of a? Draw the bifurcation diagram. For what value of a does there exist a semistable equilibrium? Draw the phase line for several values of a.

   We solve for the equilibria as $y^2 - 2y - a = 0$, which by the quadratic equation gives $y = \frac{2\pm\sqrt{4+4a}}{2}$, so $y = 1 \pm \sqrt{1+a}$. If $a < -1$, there are no equilibria. If $a = -1$, there is a single one, and it's semistable. If $a > -1$, then there are two equilibria. They're both positive if $\sqrt{1+a} < 1$, which means $-1 < a < 0$. The bifuration diagram is a plot of $y(2-y) + a = 0$, which is a parabola opening to the right.

5.

   (a) *Use basic facts about complex numbers to prove the Pythagorean identity* $\cos^2\theta + \sin^2\theta = 1$.

   (b) *Express* $\tan\theta$ *in terms of* $e^{i\theta}$.

   (c) *How many solutions are there to $z^3 = i$? Compute them.*

   (a) Let $z = e^{i\theta}$. Then $z\bar{z} = e^{i\theta}e^{-i\theta} = 1$. On the other hand, it's $(\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta) = (\cos^2\theta - \sin^2\theta) + 0i$. Comparing the real parts gives the identity in question. (This is actually a special case of the sum of angles formula you saw in lecture, with $\theta_1 = \theta$, $\theta_2 = -\theta$.

(b) We know

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}, \qquad \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}.$$

So

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = -i\frac{e^{i\theta}-e^{-i\theta}}{e^{i\theta}+e^{-i\theta}}.$$

(c) There are three, by the fundamental theorem of algebra. Suppose they are $re^{i\theta}$. Then $r^3e^{3i\theta} = i$. Comparing the moduli, it must be that $r = 1$. From the arguments, $3\theta = \pi/2 + 2\pi k$ for some integer value of k. Thus $\theta = \pi/6 + 2\pi k/3$. For $k = 0, 1, 2$ this gives the three angles $\pi/6, 5\pi/6, 9\pi/6$. The three points lie in an equilateral triangle.

6. *Use Euler's method to approximate $y(1)$ for a solution to $y' = y$ with $y(0) = 1$. What is the true value? What estimate do you get for the final value when your step size is $1/n$, and what happens when $n \to \infty$?*

I'll use a step size of 1/4.

x_k	y_k	m_k
0	1	1
1/4	5/4	5/4
1/2	25/16	25/16
3/4	125/64	125/64
1	625/256	

This is 2.441. The real value is e, which is 2.718, so we didn't do too badly.

Note that at every stage we had multiplied by 5/4, so really this was $\left(1+\frac{1}{4}\right)^4$. For large values of 4 (i.e. small step size), we get the familiar fact that

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e.$$

7.

(a) *Sketch the graph of $f(t) = 3\cos(2t-\pi)$.*

(b) *Express $2\cos(2t) + 3\sin(2t)$ in standard form.*

(c) *What is $\int e^t \sin 2t\, dt$?*

(a) Put this into Wolfram Alpha if you want a plot.

(b) Use the rule $A\cos(\omega t - \phi) = a\cos(\omega t) + b\sin(\omega t)$, where $A = \sqrt{a^2+b^2}$, $a = A\cos\phi$, $b = B\sin\phi$. We're given $a = 2$ and $b = 3$, so $A = \sqrt{13}$. Now, $2 = \sqrt{13}\cos\phi$, so $\cos\phi = 2/\sqrt{13}$. Similarly $\sin\phi = 3/\sqrt{13}$. Thus $\phi = \cos^{-1}(2/\sqrt{13})$.

$$2\cos(2t) + 3\sin(2t) = \sqrt{13}\cos(2t - \cos^{-1}(2/\sqrt{13})).$$

(c) Note that $e^t \sin 2t$ is the imaginary part of $e^{(1+2i)t}$. So its integral is the imaginary part of the integral of this function.

$$\begin{aligned}\int e^{(1+2i)t}\,dt &= \frac{1}{1+2i}e^{(1+2i)t} = \frac{1-2i}{5} \cdot e^t \cdot (\cos 2t + i \sin 2t) \\ &= \frac{1}{5}e^t\left((\cos 2t + 2\sin 2t) + (-2\cos 2t + \sin 2t)i\right)\end{aligned}$$

The imaginary part is $\frac{1}{5}e^t(-2\cos 2t+\sin 2t)$, which is our answer. But don't forget the constant of integration!