Math 18.06, r3/5
Problems #1
February 12, 2013

1. Consider the system of two linear equations $x - y = 0$, $x + y = 2$.
   (a) Express this sytem in matrix form.
   (b) What is the "row picture" for this system?
   (c) What is the "column picture"?

2. Consider the matrix $A = \begin{bmatrix} 2 & 3 & 0 \\ 4 & 5 & 1 \\ 2 & -1 & 6 \end{bmatrix}$.
   (a) What is the linear system of equations corresponding to $A\mathbf{x} = \left(\begin{smallmatrix} 3 \\ 7 \\ 5 \end{smallmatrix}\right)$?
   (b) Solve the system using elimination.
   (c) Give an LU decomposition for A.

3. (a) Write down a matrix A which projects 3-dimensional vectors onto the 2-dimensional xy-plane. What are the dimensions of A?
   (b) Write down a 2×2 matrix B which rotates vectors 90° counterclockwise, and a 3×3 matrix which rotates 90° counterclockwise around the z-axis without changing the height.
   (c) What is a 2×3 matrix C that projects vectors to the xy-plane and then rotates them 90° clockwise?
   (d) Let A be the matrix from part (a). For what 3×3 matrices D is it true that $A\mathbf{v} = A(D\mathbf{v})$ for all vectors $\mathbf{v}$?

4. Consider the matrix
$$M = \begin{bmatrix} 1 & c & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$
   Suppose you try to solve $M\mathbf{x} = \left[\begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix}\right]$ using elimination. For what value(s) of c does this fail? Interpret this in terms of the row and column pictures for the linear system.

5. Use Gauss-Jordan elimination to compute A^{-1}, where A is the matrix from the second problem. Use this to solve the system in problem 2(b) again.

6. Suppose that $\mathbf{u}$ and $\mathbf{v}$ are two vectors. Let A be the matrix whose columns are $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u} + \mathbf{v}$. Let B be the matrix whose rows are $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u} + \mathbf{v}$. Can you find a vector $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{0}$? Such that $B\mathbf{x} = \mathbf{0}$?

7. Can you find any 2×2 matrices with the following properties? Can you find all of them? (Hint: there may be none)
   (a) $A\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{v}$ for any $\mathbf{u}$ and $\mathbf{v}$.
   (b) $A\mathbf{u} \cdot A\mathbf{v} = \mathbf{u} \cdot \mathbf{v}$ for any $\mathbf{u}$ and $\mathbf{v}$.
   (c) A is a 2×2 matrix such that $A\mathbf{u}$ is a vector of length $|\mathbf{u}|$ in the direction $(1, 0)$.

8. Roughly how many addition and multiplication operations are required to compute the inverse of an $n \times n$ matrix using Gauss-Jordan elimination?