Math 18.06, r3/5/6
Problems #3
March 13, 2013

1. Suppose that P is the matrix for projection onto a subspace $V \subset \mathbb{R}^3$.
   (a) If ________________, then $P\mathbf{x} = \mathbf{x}$.
   (b) If ________________, then $P\mathbf{x} = \mathbf{0}$.
   (c) P^2 is ________________.
   (d) (T/F) If two subspaces are orthogonal, they meet only at the zero vector.
   (e) (T/F) If two subspaces meet only at the zero vector, they are orthogonal.
   (f) Describe three nonzero subspaces of $\mathbb{R}^3$, each of which is orthogonal to the other two.
2. Let $\mathbf{v} = (1, 1, 1)$ and $\mathbf{w} = (1, 2, 3)$ be two vectors in $\mathbb{R}^3$, and $V \subset \mathbb{R}^2$ be the plane they span.
   (a) Write down the matrix P for projection onto V. If you finish all the multiplications, how can you check that your answer is plausible?
   (b) Give a basis for $V^\perp$, the orthogonal complement of V. What's the matrix for projection onto $V^\perp$?
   (c) How are $P_V\mathbf{x}$ and $P_{V^\perp}\mathbf{x}$ related? Find the point in V that is closest to $(1, 0, 0)$.
   (d) We know that in this case (V a two-dimensional subspace of $\mathbb{R}^3$ spanned by $\mathbf{v}$ and $\mathbf{w}$), the orthogonal complement is spanned by $\mathbf{v} \times \mathbf{w}$. Check that this gives the same result.
3. Suppose that V and W are two orthogonal subspaces of $\mathbb{R}^n$. What are the possible values of $\dim V + \dim W$?
4. Suppose that V and W are the 1-dimensional subspaces of $\mathbb{R}^3$ spanned by $(1, 0)$ and $(1, 1)$ respectively. Let P_V and P_W be the corresponding projection matrices.
   (a) Find the product $P_V P_W$. Is this a projection map? Geometrically, what does $P_V P_W$ do to a vector?
   (b) Suppose $\mathbf{x}$ is a vector. Must $P_V\mathbf{x} + P_W\mathbf{x} = \mathbf{x}$?
5. Suppose that V and W are two subspaces of $\mathbb{R}^3$. What must be true of V and W to guarantee that $P_V\mathbf{x} + P_W\mathbf{x} = \mathbf{x}$, for all vectors $\mathbf{x}$?
6. (8.2.1-2) Let A be the triangle graph, with three vertices p_1, p_2, p_3, and three edges: $p_1 \to p_2$, $p_1 \to p_3$, $p_2 \to p_3$.
   (a) Write down the incidence matrix for A. What vectors are in the nullspace of A? What vectors are in the row space?
   (b) Write down A^T. What vectors $\mathbf{y}$ are in the nullspace? Interpret this in terms of current.
7. Suppose that $p_1, \ldots, p_n$ are vertices of a graph. What is the maximum number of edges that can be added between them without creating any cycles? Interpret this in terms of linear algebra. If $n = 4$, how many different ways can you find to add 3 edges without making a loop? (don't worry about making a directed graph)