Math 18.06, r3/5
Problems #6
April 3, 2013

1. (a) Say you try to use Cramer's rule to solve $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b}$ is equal to one of the columns of $A$. What happens?
   (b) Suppose that $A$ is a matrix with integer entries, and $\mathbf{b}$ is a vector with integer entries. Under what conditions on $A$ must the solutions to $A\mathbf{x} = \mathbf{b}$ also have integer entries? Explain this using Cramer's rule.
   (c) Compute the area of a triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$.
   (d) Consider the matrix
$$A = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 4 \end{pmatrix}.$$
   Compute $\det A$ by each of the three main methods: cofactor expansion, the "big formula", and multiplication of pivots. What are the eigenvalues and eigenvectors?

2. Suppose that $A$ is an upper-triangular matrix. What does the cofactor matrix $C$ look like? What does this tell you about $A^{-1}$?

3. Let $A_n$ be the $n \times n$ matrix shown below:
$$A_n = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 0 & 2 \end{pmatrix}$$
   Let $b_n = \det A_n$. Express $b_n$ in terms of $b_{n-1}$. Can you find a formula for $b_n$?

4. Suppose that $A$ is a $2 \times 2$ matrix with eigenvalues $\lambda_1$ and $\lambda_2$. What are the trace and determinant of $A$? What are the trace and determinant of $A^2$? Can you express these in terms of $\det A$ and $\operatorname{tr} A$?

5. Suppose that $B(t)$ is the following matrix (which depends on a parameter $t$):
$$\begin{pmatrix} t & t^2 & 1 & 2 \\ 3 & 4 & 5 & 6 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$
   For how many values of $t$ do you expect that $\det B(t) = 0$? What is $\frac{d}{dt}\det(B(t))$?

6. Given three vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in $\mathbb{R}^4$, define the "cross product" of the three vectors to be
$$\mathbf{u} \boxtimes \mathbf{v} \boxtimes \mathbf{w} = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} & \mathbf{l} \\ u_1 & u_2 & u_3 & u_4 \\ v_1 & v_2 & v_3 & v_4 \\ w_1 & w_2 & w_3 & w_4 \end{pmatrix}.$$
   Explain why $\mathbf{u} \boxtimes \mathbf{v} \boxtimes \mathbf{w}$ is perpendicular to each of the three vectors.

7. Give an example of a $2 \times 2$ matrix with both eigenvalues equal, which isn't a multiple of the identity. Geometrically, what does your matrix do to a vector? What are the eigenvectors?

8. Use the big formula to explain why $\det A = \det A^T$ for a square matrix $A$.