Math 18.06, r3/5
Problems #9 or so
May 7, 2013

1. Let
$$A = \begin{pmatrix} x & 2 \\ 2 & 1 \end{pmatrix}.$$
For what values of x is A positive-definite? For what values of x is it diagonalizable?

2. Remember that the Fibonacci sequence is defined by $F_{n+2} = F_{n+1} + F_n$.
   (a) Let $\mathbf{u}_n = (F_{n+1}, F_n)$. Write down a recurrence $\mathbf{u}_{n+1} = A\mathbf{u}_n$.
   (b) Give a formula for F_n.
   (c) If you divide one column of your matrix A by some factor, it becomes a Markov matrix. What column, and what factor? Describe the what this matrix does, and find the steady-state solution.

3. Suppose that A and B are symmetric matrices which are positive definite. Prove that AB has all eigenvalues positive. (Hint: write $AB\mathbf{x} = \lambda\mathbf{x}$ and take the dot product of both sides with $B\mathbf{x}$).

4. Let
$$A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$
be a non-diagonalizable 2×2 Jordan block. What is A^2? A^3? A^n? Compute e^{At}. Hint: use the identity
$$\sum_{n=1}^{\infty} \frac{n\lambda^{n-1}}{n!} t^n = te^{\lambda t}$$

5. Suppose A is a 2×2 matrix that isn't diagonal but has repeated eigenvalues. Prove that it can't be diagonalizable. What if A is 3×3?
   (a) Let $A = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$. Compute the SVD of A.
   (b) How can you read off the four fundamental subspaces of A from your decomposition?
   (c) Say A is any rank-1 matrix. Describe the SVD for A. Is this consistent with your answer to (a)?

6. Find the solution to $y'' - 3y' + 2y = 0$ that satisfies $y(0) = 2$ and $y'(0) = 3$. (Hint: use the variables $u(t) = y(t)$ and $v(t) = y'(t)$.

7. Prove that if A is similar to B, and B is similar to C, then A is similar to C. Conclude that if A and B are symmetric and have the same eigenvalues, they are similar.

8. Sketch the ellipse defined by $2x^2 + 2xy + 2y^2 = 1$.

9. Suppose that A and B are Markov matrices. Which of the following must also be Markov? For each, either prove it or give a counterexample. A^{-1}, $2A$, $(2A + B)/3$, $2A - B$, A^3.