Math 18.06, r3/5
Problems #3
March 8, 2013

1. *Let U be a matrix in echelon form:*

$$U = \begin{pmatrix} 1 & 2 & 0 & 4 & 5 \\ 0 & 0 & 1 & 7 & 8 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Describe a basis for each of the four fundamental subspaces. If A reduces to U under elimination, what can you say about the fundamental subspaces of A?

The first two rows give a basis for the row space. The nullspace is spanned by the special solutions corresponding to the free variables: $(-2,1,0,0,0)$, $(-4,0,-7,1,0)$, $(-5,0,-8,0,1)$.

The column space of U is spanned by $(1,0,0)$ and $(0,1,0)$, the pivot columns. The left nullspace is orthogonal to the column space, so it is spanned by $(0,0,1)$.

If A reduces to U under elimination, its row space and nullspace are the same as those for U. As for the column space, we can't say (U and A don't necessarility have the same column space). In any case the dimension of the column space is the same.

2. *Suppose that A is a 3×3 matrix. What are the possible sets of dimensions for the four fundamental subspaces? For each possibility, give an example of such a matrix and describe geometrically the corresponding transformation.*

There are four possibilities for the rank: 0, 1, 2, 3. Once we know the rank, all the dimensions are determined.

(a) Rank 0: it must be the 0 matrix. The row space and column space and both 0 dimensional, while the nullspace and left nullspace are both 3 dimensional.

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

(b) Rank 1: the dimension of the row space and column space are 1. The nullspace and left nullspace are both 2 dimensional.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

(c) Rank 2: the row space and column space are both 2 dimensional, and the nullspace and left nullspace are 1 dimensional.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

I'll just discuss the geometry for this one. A vector (x,y,z) is projected onto the xy-plane, giving $(x,y,0)$. The nullspace is the set of vectors which mapped to 0 by A, namely anything on the z axis $(0,0,z)$. The column space is set of all vectors that are in the image of A, namely the entire xy-plane. The row space is also the xy-plane (it's perpendicular to the nullspace), and the left nullspace is the z-axis.

(d) Rank 3: row space is 3 diml, as is column space. Nullspace and left nullspace are 0 dimensional.

3. *Suppose you were handed a 3×5 matrix. What procedure would you use to compute a basis for each of the four fundamental subspaces?*

   (a) The row space: you can use row reduction on A. The row operations don't change the row space, so the nonzero rows of rref are a basis.

   (b) Nullspace: use the special solutions.

   (c) Column space: either take the pivot columns (of A, not of R, but you uncover their identities via elimination), or do reduction on the transpose.

   (d) Left nullspace: you can run elimination and find special solutions for the transpose.

4. *Let A be the product*

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

   Find the four fundamental subspaces first for the factor on the right, and then for A itself. How would your answer change if the last row of the right-hand factor were replaced with 0s?

   First we find the nullspace of the right matrix. The usual special solutions stuff gives $\mathbf{v} = (0, 0, -1, 1)$ as a basis for the nullspace. The row space is spanned by the rows of rref, which are $(1, 0, 0, 0)$, $(0, 1, 0, 0)$, and $(0, 0, 1, 1)$. Note that these are orthogonal to the nullspace.

   Since the first three columns are independent, they must span all of $\mathbb{R}^3$. This means that the column space is $\mathbb{R}^3$, while the left nullspace is 0.

   On to the full thing. What's in the nullspace? There are two ways we could have $A\mathbf{v} = (BC)\mathbf{v} = \mathbf{0}$. Either $C\mathbf{v} = 0$, and $\mathbf{v} = c(0, 0, 1, 1)$ as we've already seen, or $C\mathbf{v} \neq \mathbf{0}$, but $B\mathbf{v} = 0$ anyway. In this example, this is impossible: there is nothing nonzero in the nullspace of $\mathbf{v}$, so if $C\mathbf{v} \neq \mathbf{0}$, then $BC\mathbf{v} \neq \mathbf{0}$ as well. So the full nullspace is spanned by $(0, 0, 1, 1)$. The row space is orthogonal to the nullspace, so it's the same as above.

   Now, the column space of B is all of $\mathbb{R}^3$. Since C has column space all of $\mathbb{R}^3$, so too does the product, and the left nullspace is again 0.

5. *Suppose A is a $m \times n$ matrix, and you know there is a vector $\mathbf{b}$ for which $A\mathbf{x} = \mathbf{b}$ has no solution. What does this tell you about the four fundamental subspaces?*

   This means that the column space isn't all of $\mathbb{R}^n$. In particular, $\dim C(A) < n$, and so $\dim N(A^T) \geq 1$. In particular, there is a solution to $A^T\mathbf{x} = \mathbf{0}$.

6. *Every rank 1 matrix can be written in the form $\mathbf{u}\mathbf{v}^T$ (column times row). Explain this for the rank-1 matrix I spotted at the store last weekend:*

$$\left(\begin{array}{r|r|r|r} 3\ t & 1\ T & 1/16\ c & 1/64\ qt \\ 12\ t & 4\ T & 1/4\ c & 1/16\ qt \\ 24\ t & 8\ T & 1/2\ c & 1/8\ qt \\ 48\ t & 16\ T & 1\ c & 1/4\ qt \end{array}\right)$$

   Take $\mathbf{v} = (3, 1, 1/16, 1/64)$ and $\mathbf{u} = (1, 4, 8, 16)$.