Math 18.06, r3/5/6
Problems #3
April 6, 2013

1. *Suppose that P is the matrix for projection onto a subspace $V \subset \mathbb{R}^3$.*

   (a) *If* ________________, *then* $P\mathbf{x} = \mathbf{x}$.
   If $\mathbf{x}$ is contained in V, then $P\mathbf{x} = \mathbf{x}$.

   (b) *If* ________________, *then* $P\mathbf{x} = \mathbf{0}$.
   If $\mathbf{x}$ is contained in $V^\perp$, then $P\mathbf{x} = \mathbf{0}$.

   (c) P^2 *is* ________________.
   P^2 is equal to P.

   (d) *(T/F) If two subspaces are orthogonal, they meet only at the zero vector.*
   This one's true.

   (e) *(T/F) If two subspaces meet only at the zero vector, they are orthogonal.*
   F: the spans of $(1,0)$, and $(1,1)$ in $\mathbb{R}^2$ meet only at $\mathbf{0}$, but they are not orthogonal.

   (f) *Describe three nonzero subspaces of $\mathbb{R}^3$, each of which is orthogonal to the other two.*
   Use the three coordinate axes!

2. *Let $\mathbf{v} = (1,1,1)$ and $\mathbf{w} = (1,2,3)$ be two vectors in $\mathbb{R}^3$, and $V \subset \mathbb{R}^2$ be the plane they span.*

   (a) *Write down the matrix P for projection onto V. If you finish all the multiplications, how can you check that your answer is plausible?*
   This is just a matter of using the formula. Here

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix},$$

so

$$\begin{aligned}
P = A(A^TA)^{-1}A^T &= \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} \left(\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} \right)^{-1} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} \left(\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} \right)^{-1} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 3 & 6 \\ 6 & 14 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix} \\
&= \cdots = \frac{1}{6} \begin{pmatrix} 5 & 2 & -1 \\ 2 & 2 & 2 \\ -1 & 2 & 5 \end{pmatrix}.
\end{aligned}$$

A way to check this is to plug in $(1,1,1)$ or $(1,2,3)$ and see that P gives back the same vector you put in. I tried it; it does. (This doesn't guarantee that all the calculations are right, but it is pretty strong evidence.)

(b) *Give a basis for $V^\perp$, the orthogonal complement of V. What's the matrix for projection onto $V^\perp$?*

V has dimension 2, so $V^\perp$ is going to have dimension 1. It's the nullspace of A, which is the nullspace of

$$A^T = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix}.$$

Go through elimination, and you get a single special solution $(1, -2, 1)$. The projection is just onto a line now, so we get

$$P = \frac{\mathbf{a}\mathbf{a}^T}{\mathbf{a}^T\mathbf{a}} = \frac{1}{6}\begin{pmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{pmatrix}$$

(c) *How are $P_V\mathbf{x}$ and $P_{V^\perp}\mathbf{x}$ related? Find the point in V that is closest to $(1, 0, 0)$.*

This is $P_V\mathbf{x} = (5/6, -1/3, 1/6)$. Observe that $P_V\mathbf{x} + P_{V^\perp}\mathbf{x} = \mathbf{x}$; this is analogous to being able to recover a vector (x, y, z) by adding its projection to the xy-plane $(x, y, 0)$ to its projection to the z-axis, $(0, 0, z)$.

(d) *We know that in this case (V a two-dimensional subspace of $\mathbb{R}^3$ spanned by $\mathbf{v}$ and $\mathbf{w}$), the orthogonal complement is spanned by $\mathbf{v} \times \mathbf{w}$. Check that this gives the same result.*

3. *Suppose that V and W are two orthogonal subspaces of $\mathbb{R}^n$. What are the possible values of* $\dim V + \dim W$*?*

If $\dim V + \dim W > n$, we know by an old homework problem that V and W must have a nonzero vector in common. But this is impossible for orthogonal subspaces: we need $\langle \mathbf{v}, \mathbf{w} \rangle = 0$ for $\mathbf{v} \in V$ and $\mathbf{w} \in W$, but if $\mathbf{y}$ is in both this would read $\langle \mathbf{y}, \mathbf{y} \rangle = 0$, which is impossible.

4. *Suppose that V and W are the 1-dimensional subspaces of $\mathbb{R}^3$ spanned by $(1, 0)$ and $(1, 1)$ respectively. Let P_V and P_W be the corresponding projection matrices.*

(a) *Find the product $P_V P_W$. Is this a projection map? Geometrically, what does $P_V P_W$ do to a vector?*

Clearly $P_V = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, and the formula gives $P_W = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.

Then $P_V P_W = \begin{pmatrix} 1/2 & 1/2 \\ 0 & 0 \end{pmatrix}$. This can't be a projection matrix; $P^2 \neq P$.

(b) *Suppose $\mathbf{x}$ is a vector. Must $P_V\mathbf{x} + P_W\mathbf{x} = \mathbf{x}$?*

Nope, not in general. This would work if V and W were orthogonal to each other (as in 2c), but they aren't.

5. *Suppose that V and W are two subspaces of $\mathbb{R}^3$. What must be true of V and W to guarantee that $P_V\mathbf{x} + P_W\mathbf{x} = \mathbf{x}$, for all vectors $\mathbf{x}$?*

6. (8.2.1-2) *Let A be the triangle graph, with three vertices p_1, p_2, p_3, and three edges: $p_1 \to p_2$, $p_1 \to p_3$, $p_2 \to p_3$.*

(a) *Write down the incidence matrix for A. What vectors are in the nullspace of A? What vectors are in the row space?*

It's $A = \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}$. Rref for the matrix is $R = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$. The row space is spanned by these two rows, while the nullspace is spanned by the special solution $(1, 1, 1)$.

(b) *Write down A^T. What vectors $\mathbf{y}$ are in the nullspace? Interpret this in terms of current.*

We have $A^T = \begin{pmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1-1 \end{pmatrix}$. The nullspace is again spanned by $(1, 1, 1)$. This corresponds to a current of 1 along each of the three edges.

7. *Suppose that $p_1, \dots, p_n$ are vertices of a graph. What is the maximum number of edges that can be added between them without creating any cycles? Interpret this in terms of linear algebra. If $n = 4$, how many different ways can you find to add 3 edges without making a loop? (don't worry about making a directed graph)*

   A cycle comes from dependent rows, and there can be at most 3 of these (since we know there is a nontrivial nullspace for the incidence matrix).

   There are 16 ways to add edges to four nodes: either arrange the four in a row and connect them in a line, or connect all three to a given one. There are $4! = 24$ ways to line them up, but forwards and backwards give the same graph, so divide by 2 to get 12. There are 4 possibilities for the other type, giving a total of 16.