Math 18.06, r3/5
Problems #5
March 19, 2013

1. (a) *$A^T A$ is invertible if ______________.*

   $A^T A$ is invertible if the columns of A are independent (see page 211 in the book).

   (b) *Say A is a square orthogonal matrix. Must A^T be as well? What if A is not square?*

   Yes indeed, if it's square: orthogonality means that $A^T A = I$, so $A^T = A^{-1}$. We know that this implies that $AA^T = I$ as well, i.e. $(A^T)^T(A^T) = I$, which says that A^T is orthogonal. If it's not square, this doesn't work, eg

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

   (c) *Suppose $A = LDU$. What are the determinants of each of the three factors on the right?*

   This implies that $\det A = \det L \cdot \det D \cdot \det U$. L is lower-triangular with 1s on the diagonal, so $\det L = 1$. Similarly $\det U = 1$. D has the pivots on the diagonal, so its determinant is the product of the pivots, i.e. $\det A$. Everything is consistent.

   (d) *What about $A = QR$?*

   The determinant of an orthogonal matrix is ± 1, since $\det A^T = \det A$, and so $1 = \det(I) = \det(A^T A) = \det(A)^2$. It must be that $\det R = \pm \det A$, with the sign depending on whether $\det Q$ is $+1$ or -1.

   (e) *Give an example of a 4×4 matrix A for which $\det A$ is not just (product of downward diagonals) - (product of upward diagonals). Hint: you can use a permutation matrix.*

   Use

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

   The product along any diagonal is 0, but the determinant is -1.

2. *Find a best fit parabola $y = ax^2 + bx + c$ through the four points $(-1, 0)$, $(0, 0)$, $(1, 0)$, $(2, 1)$ (or at least explain how you would). Could you find a best fit curve of the form $y = a + be^{cx}$ through these points?*

   For a perfect fit, we'd solve the four equations $c = 1$, $a + b + c = 1$, $4a + 2b + c = 5$, $9a + 3b + c = 8$, i.e. $A\mathbf{x} = \mathbf{b}$ with

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

There's no solution, so we solve $(A^T A)\hat{\mathbf{x}} = A^T\mathbf{b}$ instead.

$$A^T A = \begin{pmatrix} 1 & 0 & 1 & 4 \\ -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 18 & 8 & 6 \\ 8 & 6 & 2 \\ 6 & 2 & 4 \end{pmatrix}$$

$$A^T\mathbf{b} = \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}$$

$$\hat{\mathbf{x}} = \frac{1}{20}\begin{pmatrix} 5 \\ 1 \\ -3 \end{pmatrix}$$

So the polynomial is $f(x) = \frac{1}{4}x^2 + \frac{1}{20}x - \frac{3}{20}$. This has

$$f(-1) = \frac{1}{20}, \quad f(0) = -\frac{3}{20}, \quad f(1) = \frac{3}{20}, \quad f(2) = \frac{19}{20}.$$

We did a pretty good job interpolating.

As for the exponential fit: an exact solution would have $a + be^{-c} = 0$, $a + b = 0$, $a + be^{c} = 0$, and $a + be^{2c} = 0$. These aren't linear equations (well, except the second one). It's not $A\mathbf{x} = \mathbf{b}$, and there's nothing we can do.

3. *Let A be the matrix*

$$A = \begin{pmatrix} 1 & 0 \\ 2 & -4 \\ 3 & 2 \end{pmatrix}.$$

What could you add as the third column to get a matrix with all columns orthogonal to each other? What else do you have to do to get an orthogonal matrix?

Anything orthogonal to both of the existing columns will do: we need something in the nullspace of

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & -4 & 2 \end{pmatrix}.$$

One such is $(-8, 1, 2)$. To make it an orthogonal matrix, divide each column by its norm.

4. *Find an orthonormal basis for the subspace of $\mathbb{R}^3$ defined by $x + 2y + 3z = 0$. Once you have an answer, how can you check it?*

First we find a basis. This is the nullspace of the matrix $M = (1\ 2\ 3)$, and two special solutions are $\mathbf{a} = (-2\ 1\ 0)$ and $\mathbf{b} = (-3\ 0\ 1)$.

Our first basis vector is $\mathbf{A} = (-2\ 1\ 0)$ (we still need to normalize it at the end). The second is

$$\mathbf{B} = \mathbf{b} - \frac{\mathbf{A}^T\mathbf{b}}{\mathbf{A}^T\mathbf{A}}\mathbf{A} = \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} - \frac{6}{5}\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -3/5 \\ -6/5 \\ 1 \end{pmatrix}$$

Notice that $\mathbf{B}$ is orthogonal to $\mathbf{A}$, as we expect. So we just need to normalize both of these.

$$\mathbf{q}_1 = \frac{\mathbf{A}}{|\mathbf{A}|} = \frac{1}{5}\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \qquad \mathbf{q}_2 = \frac{\mathbf{B}}{|\mathbf{B}|} = \frac{1}{\sqrt{70}}\begin{pmatrix} -3 \\ -6 \\ 5 \end{pmatrix}$$

Things to check: these both have length 1, they're orthogonal to each other, and they both actually lie in the space in question. Everything works out in this case.

5. *Find the QR decomposition for* $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

   We have to do Gram-Schmidt for the columns. Use $\mathbf{a} = (1\ 3)$ and $\mathbf{b} = (2\ 4)$. We have $\mathbf{A} = (1\ 3)$ and $\mathbf{B} = \mathbf{b} - \frac{\mathbf{A}^T\mathbf{b}}{\mathbf{A}^T\mathbf{A}}\mathbf{A} = (3/5, -1/5)$. Normalizing these and putting them as the columns of a matrix:

$$Q = \begin{pmatrix} \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} & \frac{-1}{\sqrt{10}} \end{pmatrix}$$

   We know $R = Q^T A$, so

$$R = \begin{pmatrix} \sqrt{10} & \frac{14}{\sqrt{10}} \\ 0 & \frac{2}{\sqrt{10}} \end{pmatrix}.$$

   The decomposition is

$$A = \begin{pmatrix} \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} & \frac{-1}{\sqrt{10}} \end{pmatrix} \begin{pmatrix} \sqrt{10} & \frac{14}{\sqrt{10}} \\ 0 & \frac{2}{\sqrt{10}} \end{pmatrix}.$$

6. (a) *Let* $A = \begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & -3 & 0 \end{pmatrix}$. *Check that* $\det A = 0$.

   Go through the elimination process and you get a 0 pivot – so the determinant is 0.

   (b) *Suppose A is any skew-symmetric matrix, i.e.* $A^T = -A$. *Show that if n is odd, then* $\det A = 0$. *What can you say if n is even?*

   We always have $\det(A) = \det(-A^T) = (-1)^n \det(A^T) = (-1)^n \det(A)$. If n is odd, this says $\det(A) = -\det(A)$ and so $\det(A) = 0$. If n is even, it says $\det(A) = \det(A)$, which doesn't tell us much of anything. It turns out that if n is even this determinant is always a perfect square (and in particular is positive). You can read about this on Wiki: `http://en.wikipedia.org/wiki/Pfaffian`.

7. (a) *Let*

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 3 & 4 \end{pmatrix}.$$

   What is $\det A$*?*

   Elimination runs

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 3 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -2 \end{pmatrix}.$$

   The determinant is -2.

   (b) *Suppose that*

$$M = \left(\begin{array}{c|c} A & B \\ \hline 0 & C \end{array}\right)$$

is any 4×4 *matrix composed of four* 2×2 *blocks. Show that* $\det M = \det A\cdot\det C$. *What happens if*

$$M = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)?$$

After elimination, the pivots in the first two rows agree with the pivots of A, and the pivots in the last two rows agree with the pivots of C. So it works.

You might be tempted to guess that $\det M = \det A \det D - \det B \det C$ in the latter case, but it doesn't.

$$M = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right).$$

Here $\det M = -1$, but each of the 2×2 submatrices has determinant 0.