Math 18.06, r3/5
Problems #9 or so
May 7, 2013

1. *Let*

$$A = \begin{pmatrix} x & 2 \\ 2 & 1 \end{pmatrix}.$$

For what values of x is A positive-definite? For what values of x is it diagonalizable?

Positive-definite requires the top left determinants to be positive. This means $x > 0$ and $x - 4 > 0$ respectively. So we're good as long as $x > 4$.

It's symmetric, so always diagonalizable! The thing to watch out for in similar problems, though, is for the value of x that would give a repeated eigenvalue.

2. *Remember that the Fibonacci sequence is defined by $F_{n+2} = F_{n+1} + F_n$.*

   (a) *Let $\mathbf{u}_n = (F_{n+1}, F_n)$. Write down a recurrence $\mathbf{u}_{n+1} = A\mathbf{u}_n$.*
   We have

$$\begin{pmatrix} F_{n+2} \\ F_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_{n+1} \\ F_{n+2} \end{pmatrix}.$$

   (b) *Give a formula for F_n.*
   The eigenvalues are $\lambda_1 = \frac{1+\sqrt{5}}{2}$ and $\lambda_2 = \frac{1-\sqrt{5}}{2}$. The eigenvectors are $(\lambda_1, 1)$ and $(\lambda_2, 1)$. Our initial F_0 is $(1, 0)$, which is $(\mathbf{x}_1 - \mathbf{x}_2)/(\lambda_1 - \lambda_2)$. Then

$$\mathbf{u}_n = \frac{(\lambda_1)^n \mathbf{x}_1 - (\lambda_2)^n \mathbf{x}_2}{\lambda_1 - \lambda_2}.$$

   (c) *If you divide one column of your matrix A by some factor, it becomes a Markov matrix. What column, and what factor? Describe the what this matrix does, and find the steady-state solution.*

3. *Suppose that A and B are symmetric matrices which are positive definite. Prove that AB has all eigenvalues positive. (Hint: write $AB\mathbf{x} = \lambda\mathbf{x}$ and take the dot product of both sides with $B\mathbf{x}$).*

Say $AB\mathbf{x} = \lambda\mathbf{x}$. Then $\lambda \mathbf{x}^T B^T \mathbf{x} = \mathbf{x}^T B^T AB\mathbf{x}$. This gives

$$\lambda = \frac{\mathbf{x}^T B^T AB\mathbf{x}}{\mathbf{x}^T B^T \mathbf{x}} = \frac{(B\mathbf{x})^T A (B\mathbf{x})}{\mathbf{x}^T B^T \mathbf{x}}.$$

Both top and bottom are greater than 0 since A and B respectively are positive definite.

4. *Let*

$$A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$

be a non-diagonalizable 2×2 Jordan block. What is A^2? A^3? A^n? Compute e^{At}. Hint: use the identity

$$\sum_{n=1}^{\infty} \frac{n\lambda^{n-1}}{n!} t^n = te^{\lambda t}$$

Computing the first few, we find

$$A^2 = \begin{pmatrix} \lambda^2 & 2\lambda \\ 0 & \lambda^2 \end{pmatrix}, \quad A^3 = \begin{pmatrix} \lambda^3 & 3\lambda^2 \\ 0 & \lambda^3 \end{pmatrix}, \quad A^n = \begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix}$$

Then

$$\begin{aligned} e^{At} &= I + At + \frac{1}{2}(At)^2 + \frac{1}{6}(At)^3 + \cdots \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} + \frac{1}{2}\begin{pmatrix} \lambda^2 & 2\lambda \\ 0 & \lambda^2 \end{pmatrix} + \frac{1}{6}\begin{pmatrix} \lambda^2 & 2\lambda \\ 0 & \lambda^2 \end{pmatrix} \end{aligned}$$

Collecting the terms in each entry and summing the series (using the hint for the top-right one), this gives

$$e^{At} = \begin{pmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix}$$

5. *Suppose A is a 2×2 matrix that isn't diagonal but has repeated eigenvalues. Prove that it can't be diagonalizable. What if A is 3×3?*

If A has repeated eigenvalues, it must actually have all eigenvalues equal. If it can be diagonalized, it's $S(\lambda I)S^{-1} = \lambda I$ – the only diagonalizable 2×2 matrices with repeated eigenvalues are matrices which are already diagonal! This certainly isn't true in the 3×3 case.

(a) *Let $A = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$. Compute the SVD of A.*

First we need the eigenvectors of A^TA. We have

$$A^TA = \begin{pmatrix} 5 & 5 \\ 5 & 5 \end{pmatrix}, \qquad AA^T = \begin{pmatrix} 2 & 4 \\ 4 & 8 \end{pmatrix}.$$

The eigenvectors of the first of these are $\mathbf{v}_1 = (1,1)/\sqrt{2}$ with $\lambda = 10$ and $\mathbf{v}_2 = (1,-1)/\sqrt{2}$ with $\lambda = 0$. We $Av_1 = (2,4)$, so $\mathbf{u}_1 = A\mathbf{v}_1/\sigma_1 = (2,4)/\sqrt{2}$. The unit vector in this direction is $(2,4)/\sqrt{20} = (1,2)/\sqrt{5}$. So we have $\mathbf{u}_1 = (1,2)/\sqrt{5}$ and $\sigma_1 = \sqrt{10}$.

The decomposition is

$$A = \left(\frac{1}{\sqrt{5}}\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}\right)\begin{pmatrix} \sqrt{10} & 0 \\ 0 & 0 \end{pmatrix}\left(\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\right)$$

(b) *How can you read off the four fundamental subspaces of A from your decomposition?*

The row space is the first column of V, which is $(1,1)$. The nullspace is the other column of V, which is $(1,-1)$. The column space is the first column of U, which is $(1,2)$. The left nullspace is the second column of U, which is $(-2,1)$.

(c) *Say A is any rank-1 matrix. Describe the SVD for A. Is this consistent with your answer to (a)?*

If A is rank 1, we can write $A = \mathbf{u}\mathbf{v}^T$ where $\mathbf{u}$ and $\mathbf{v}$ are vectors. Then $A^TA = \mathbf{v}\mathbf{u}^T\mathbf{u}\mathbf{v}^T = |\mathbf{u}|^2\,\mathbf{v}\mathbf{v}^T$. This has rank 1, and an eigenvector is $\mathbf{v}$, with $A^TA\mathbf{v} = |\mathbf{u}|^2\,\mathbf{v}\mathbf{v}^T\mathbf{v} = |\mathbf{u}|^2\,|\mathbf{v}|^2\,\mathbf{v}$. So the eigenvalue is $\sigma_1^2 = |\mathbf{u}|^2\,|\mathbf{v}|^2$, and we have $\sigma_1 = |\mathbf{u}|\,|\mathbf{v}|$. The unit eigenvector (that we want for V) is $\mathbf{v}/\,|\mathbf{v}|$.

On the other hand, $AA^T = \mathbf{u}\mathbf{v}^T\mathbf{v}\mathbf{u}^T = |\mathbf{v}|^2\,\mathbf{u}\mathbf{u}^T$. This has eigenvector $\mathbf{u}$, with eigenvalue $|\mathbf{u}|^2\,|\mathbf{v}|^2$. The unit eigenvector is $|\mathbf{u}|\,/\mathbf{u}$.

So U should have first column $\mathbf{u}$, and the other columns should fill out an orthonormal basis. Similarly V has first column $\mathbf{v}$, and the other columns fill out a basis. σ should have a single entry of $|\mathbf{u}|\,|\mathbf{v}|$ in the upper-left.

This checks out with the example. Here $A = \left(\begin{smallmatrix}1\\2\end{smallmatrix}\right)\left(\begin{smallmatrix}1 & 1\end{smallmatrix}\right)$. Everything matches the description above.

6. *Find the solution to $y'' - 3y' + 2y = 0$ that satisfies $y(0) = 2$ and $y'(0) = 3$. (Hint: use the variables $u(t) = y(t)$ and $v(t) = y'(t)$.*

We have

$$\begin{pmatrix} u'(t) \\ v'(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}$$

The eigenvectors are $(1,1)$ with $\lambda = 1$ and $(1,2)$ with $\lambda = 2$. So we know the solutions are

$$\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = c_1 e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Now use the initial conditions to solve for the constants. Both are 1.

There's another way to write this out, which is really nothing more than solving for the constants. The solution is $\mathbf{u}(t) = e^{At}\mathbf{u}(0)$. We have $\mathbf{u}(0) = (2,3)$. Next order of business is to compute e^{At}. Then

$$A = S\Lambda S^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

so

$$e^{At} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}.$$

This gives

$$\mathbf{u}(t) = \begin{pmatrix} e^t + e^{2t} \\ e^t + 2e^{2t} \end{pmatrix}.$$

The solution is the top row, $y(t) = e^t + e^{2t}$.

7. *Prove that if A is similar to B, and B is similar to C, then A is similar to C. Conclude that if A and B are symmetric and have the same eigenvalues, they are similar.*

This means we can write $A = M^{-1}BM$ and $B = N^{-1}CN$. The n$A = M^{-1}N^{-1}CNM = (NM)^{-1}C(NM) = P^{-1}CP$, where $P = NM$. This means that A and C are similar.

If A and B are symmetric with the same eigenvalues, both are diagonalizable (because symmetric), and similar to the same diagonal matrix. By the above this means that they are similar to each other.

8. *Sketch the ellipse defined by $2x^2 + 2xy + 2y^2 = 1$.*

This is $\mathbf{x}^T A\mathbf{x} = 1$, where $A = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right)$. The eigenvectors are $(1,-1)$ with $\lambda = 1$ and $(1,1)$ with $\lambda = 3$. The ellipse has these as its axes, with lengths 1 and $\sqrt{3}$.

9. *Suppose that A and B are Markov matrices. Which of the following must also be Markov? For each, either prove it or give a counterexample. A^{-1}, $2A$, $(2A+B)/3$, $2A - B$, A^3.*

A^{-1} doesn't have to be Markov. An example is $A = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$, since A^{-1} has negative entries.

$2A$ can't be Markov. The columns will sum to 2, not 1; this is the case even when A is the identity matrix.

$(2A + B)/3$ must be Markov. Let $\mathbf{1}$ stand for the vector $(1, \ldots, 1)^T$. The fact that A and B are Markov means that $A^T\mathbf{1} = \mathbf{1}$ and $B^T\mathbf{1} = \mathbf{1}$. Then $((2A + B)/3)^T\mathbf{1} = (2A^T)/3\mathbf{1} + B^T/3\mathbf{1} = 2\mathbf{1}/3 + \mathbf{1}/3 = \mathbf{1}$. We also need to know that the entries of this matrix are positive, which they are, since they're a sum of positive numbers.

$2A - B$ doesn't have to be Markov (though it could be). The columns sum to 1, but maybe the entries aren't positive:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

At last, A^3 must be. Clearly the entries are all positive (think about what happens when you multiply it out). We have $(A^3)^T\mathbf{1} = (A^T)^3\mathbf{1} = \mathbf{1}$, since A itself is Markov. This means that the columns sum to 1.