Math 121 (Lesieutre); §4.2: Rational functions; September 1, 2017

1. *What is a rational function?*

It's a function of the form $\dfrac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are both polynomials. In other words, something like this:

$$r(x) = \frac{3x^2 - 2x + 7}{x^3 + x - 15}.$$

2. *Find the domains of each of the following rational functions:*

$$r_1(x) = \frac{2}{x-4}, \qquad r_2(x) = \frac{1}{x^2 - 25}, \qquad r_3(x) = \frac{5}{x^2+3}, \qquad r_4(x) = \frac{x^2-1}{x^2+3x+2}.$$

The domain is the set of all x where the function is defined. I usually find it easier to think of what's *not* in the domain: for what values of x can we not use the formula? For a rational function, the x not in the domain are the ones that make the denominator be 0.

For $r_1(x)$, only $x = 4$ is a problem: the domain is $x \neq 4$.

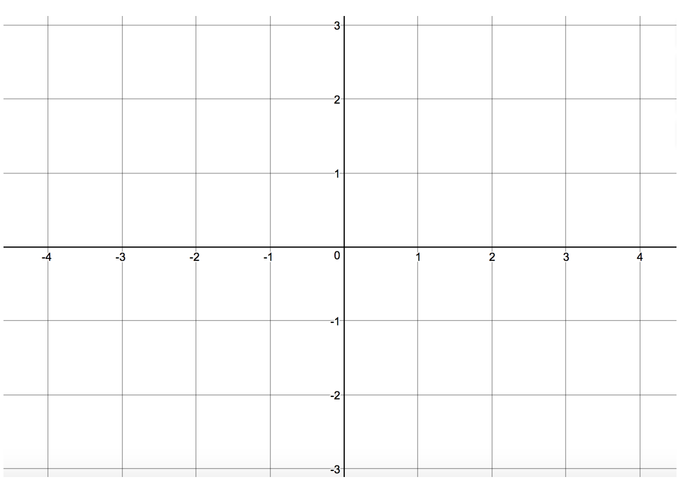
For $r_2(x)$, we're in trouble if $x^2 - 25 = 0$. This factors as $(x-5)(x+5)$, and so the domain is $x \neq 5$, $x \neq -5$.

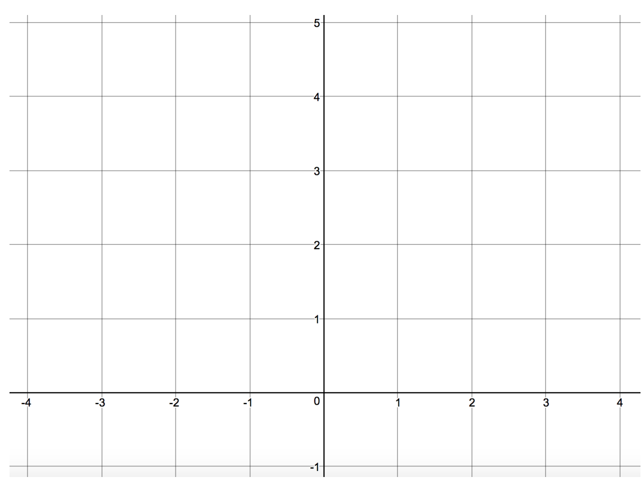
For $r_3(x)$, we're OK: $x^2 + 3$ is always positive, no matter what x i, and so there is no danger of dividing by 0. The domain is all real numbers.

For $r_4(x)$, again the issue is that $x^2 + 3x + 2 = (x+1)(x+2)$ could be 0. The domain is $x \neq -1$, $x \neq -2$.

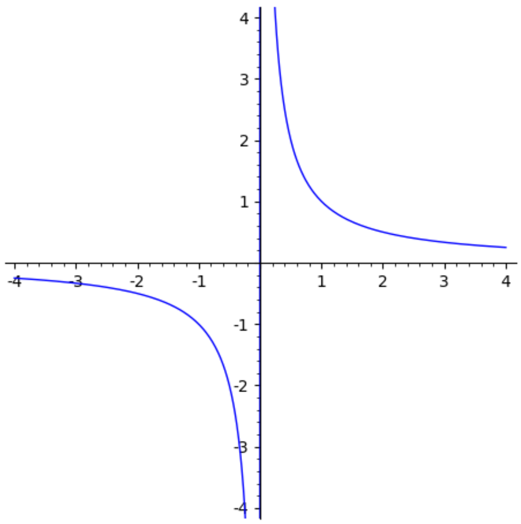
3. *Sketch the graphs of these two important rational functions, using the graph paper below.*

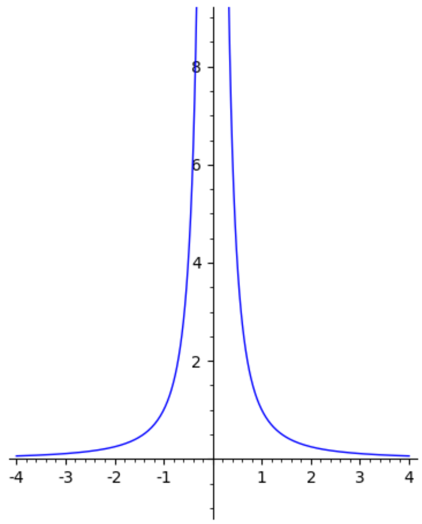
$$r_1(x) = \frac{1}{x}, \qquad r_2(x) = \frac{1}{x^2}.$$





To sketch the plot, just plug in a few x values and graph the corresponding points $(x, r(x))$. The graphs should come out like this:



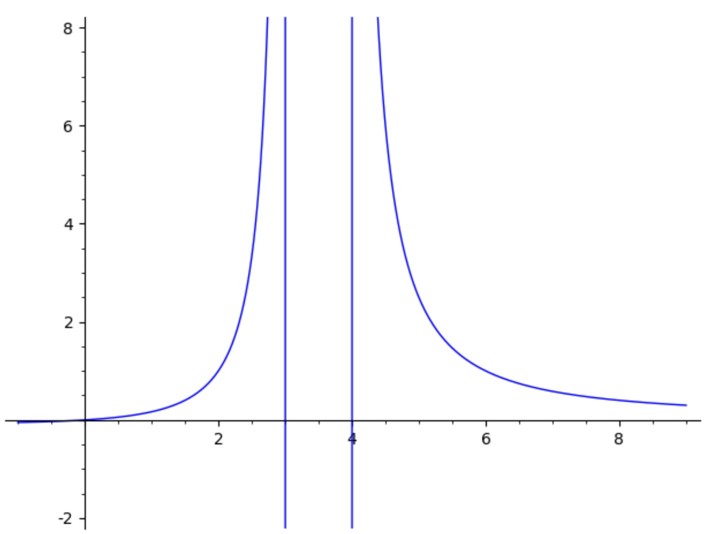


4. *How can you tell when a rational function has a vertical asymptote? Find the vertical asymptotes of the following rational functions:*

$$r_1(x) = \frac{x}{x^2 - 7x + 12}, \qquad r_2(x) = \frac{x^2 - 9}{x^2 - 9x + 18}.$$

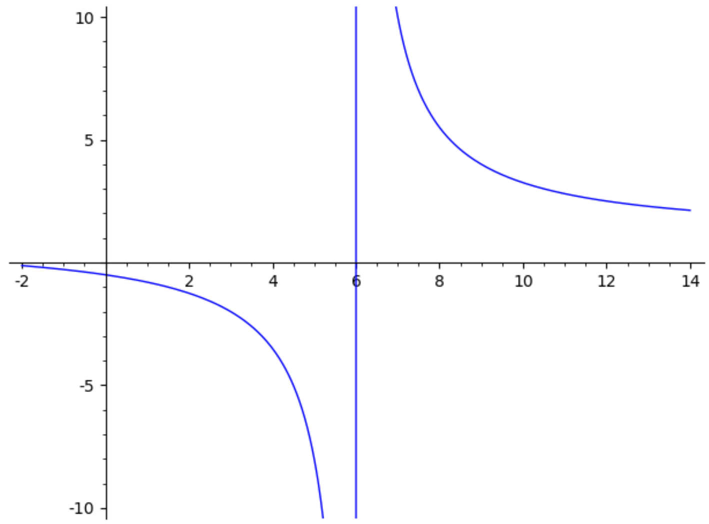
This is pretty similar to the "find the domain"-type questions. You get a vertical asymptote whenever the denominator is 0, and to figure out when that happens, you need to factor the polynomial. In this case,

For $r_1(x)$, the denominator is $x^2 - 7x + 12 = (x - 3)(x - 4)$. There ought to be vertical asymptotes at $x = 3$ and $x = 4$. Let's confirm that with a graph...



Looks good.

For $r_2(x)$, the denominator is $x^2 - 9x + 18 = (x-3)(x-6)$. You might think there are vertical asymptotes at $x = 3$ and $x = 6$. Here's the graph...



We see the asymptote at $x = 6$, but not at $x = 3$... what gives? Well, think about the numerator too:

$$r_2(x) = \frac{x^2 - 9}{x^2 - 9x + 18} = \frac{(x-3)(x+3)}{(x-3)(x-6)} = \frac{x+3}{x-6}.$$

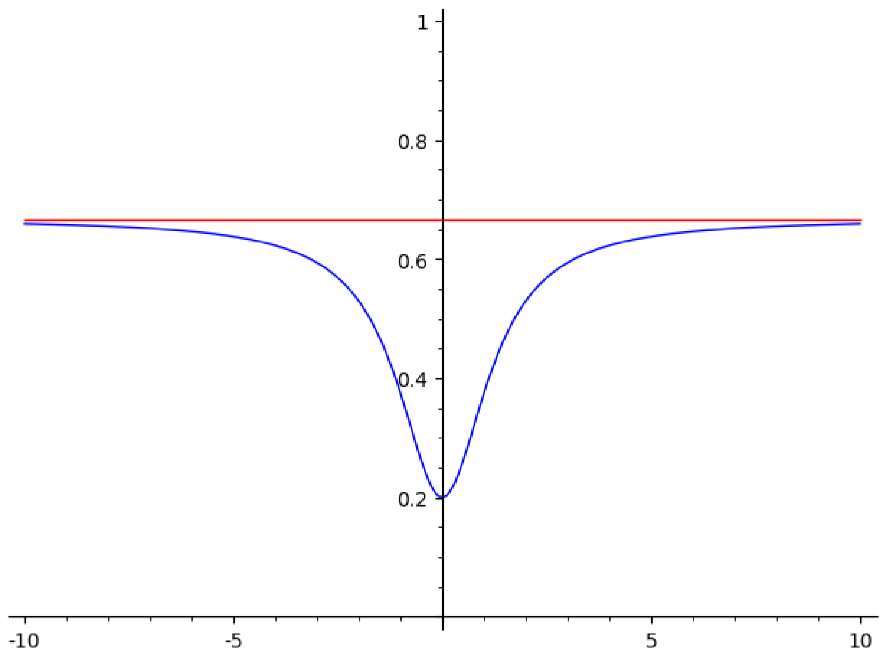
After simplifying, you no longer have an $x-6$ in the denominator. Always be sure to simplify first!

This goes for both "find the domain" and "find the vertical asymptotes": make sure the denominator is really 0, after you make all possible cancellations. If the numerator is 0 too, you can't be sure what's going to happen unless you cancel. Maybe there's an asymptote, maybe not.

5. *Finding Horizontal or Oblique asymptotes: These asymptotes describe the "long term" behavior of a function. What happens depends on how the degrees of the numerator and denominator compare...*

a) *Case 1: Degree of the numerator is less than the degree of the denominator:* $r_1(x) = \frac{1}{x}$
In this case, as $x \to -\infty$ or $x \to +\infty$, the function goes to 0. We graphed this one earlier today. In this case, $y = 0$ is a horizontal asymptote.

b) *Case 2: Degree of the numerator equals the degree of the denominator:* $r_2(x) = \frac{2x^2+1}{3x^2+5}$
In this case, as $x \to -\infty$ or $x \to +\infty$, the function goes to the ratio between the leading coefficient of the numerator, and the leading coefficient of the denominator. In the example above, that's $\frac{2}{3}$. Here's a graph to show you what's happening. The blue is the graph of the rational function in question; the red is the graph of the asymptote $y = 2/3$.

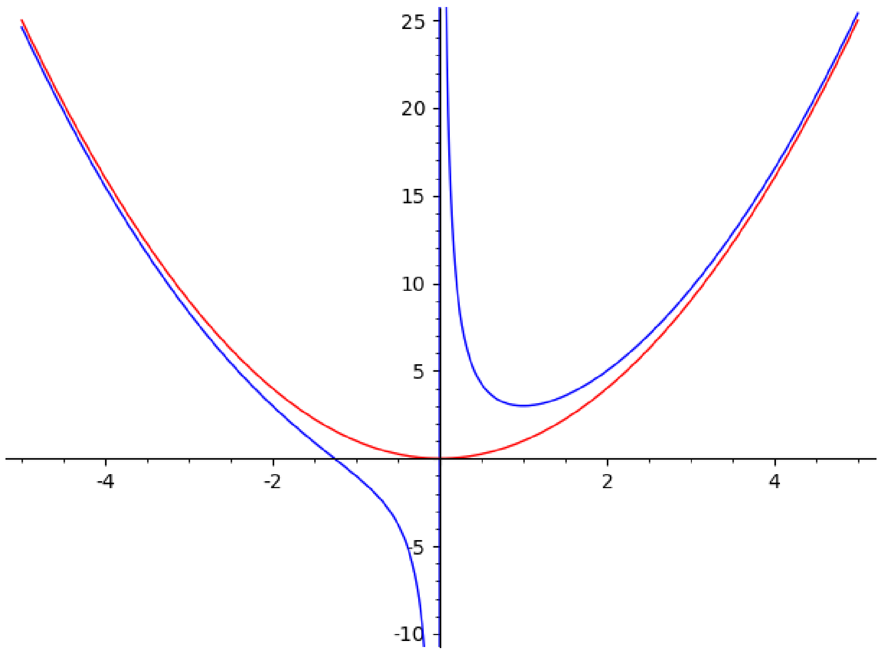


Make sure to remember this if you take Math 180/181...it comes up all the time!

c) *Case 3: Degree of the numerator is greater than the degree of the denominator:* $r_3(x) = \frac{x^3+2}{x}$

In this case, there is no horizontal asymptote. Instead, what happens is that the function goes to ∞. For large x, you can get a good approximation of the function by ignoring everything except the leading term of the numerator ad the leading term of the denominator. For the example, that would be $r_3(x) \approx \frac{x^3}{x} = x^2$.

Here's a plot showing both the function (blue) and the approximation (red), which gets better and better the further out you go.

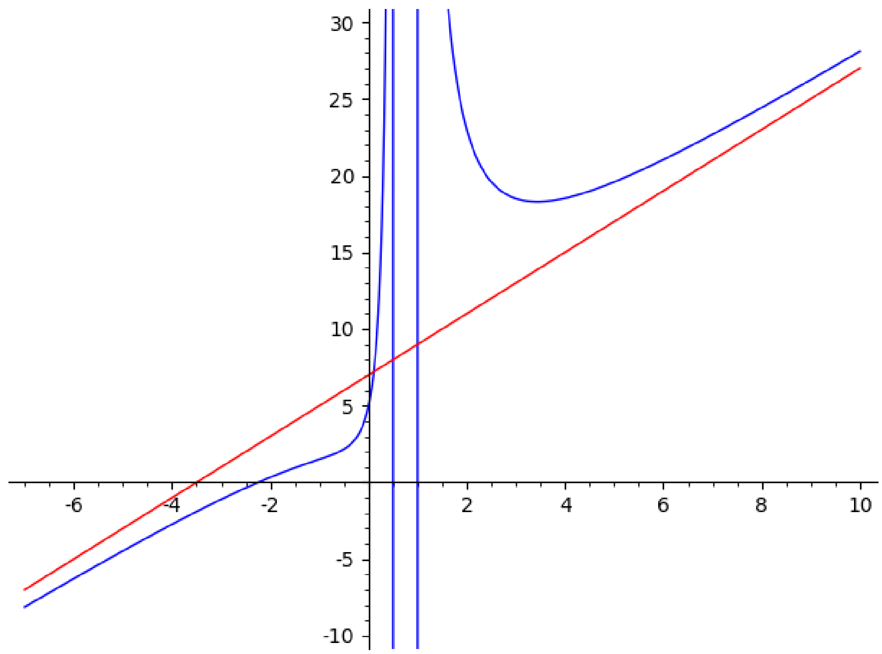


d) *Subcase 3': If the degree of the numerator is exactly one more than the degree of the denominator there is no horizontal asymptote, but you do get an Oblique (or Slant) asymptote.* Why? It's similar to the previous one. To get an approximation your best bet is to do polynomial long division. The result will be the formula for a line, plus a rational function that goes to 0. The line will be the "slant asymptote". This is more clear if we think about an example.

$$r_4(x) = \frac{4x^3 + 8x^2 + 5}{2x^2 - 3x + 1} = (\text{a lot of algebra}) = (2x + 7) + \frac{19x - 2}{2x^2 - 3x + 1}.$$

When x is big, the last part goes to 0, and so our function is very close to $2x + 7$, which is the slant asymptote.

Once again, here's a plot showing both the function (blue) and the oblique asymptote we came up with (red).



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