Math 121 (Lesieutre); §4.3; September 6, 2017

The steps for graphing a rational function:

1. Factor the numerator and denominator, and write the function in lowest terms.
2. Set the numerator equal to zero to find the x-intercepts (don't forget about multiplicity!)
3. Plug in $x = 0$ to find the y-intercept.
4. Set the denominator equal to zero to find the vertical asymptotes.
5. Find the horizontal asymptotes, depending on the degree of numerator and denominator:
   (a) If degree of numerator is less than degree of denominator, $y = 0$ is asymptote.
   (b) If degree of num. = degree of denom., then $y = \frac{\text{leading coef of numerator}}{\text{leading coef of denominator}}$ is asymptote.
   (c) If degree of numerator is greater than degree of denominator, there is no horizontal asymptote.
6. Split the x-axis into intervals, breaking it up whereever the numerator or denominator is 0. For each interval, figure out if the graph is above or below the axis in that interval, by plugging in a test number.

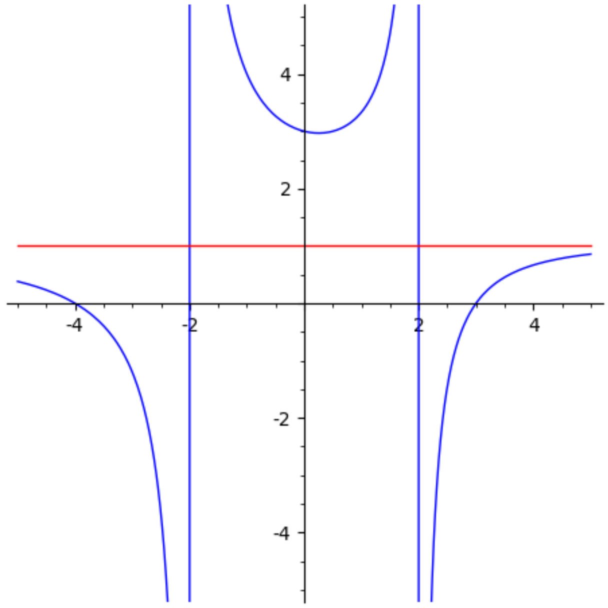
1. $f(x) = \dfrac{x^2 + x - 12}{x^2 - 4}$

Let's run through the steps.

1. The numerator is $(x+4)(x-3)$; the denominator is $(x-2)(x+2)$. Nothing cancels.
2. The zeroes are at $x = -4$ and $x = 3$.
3. The y-intercept is 3.
4. The vertical asymptotes are at 2 and -2.
5. The numerator and denominator have equal degree, so there is a horizontal asymptote. For us, it's $\frac{1}{1} = 1$.
6. Now we split the x-axis into the intervals $(-\infty, -4)$, $(-4, -2)$, $(-2, 2)$, $(2, 3)$, $(3, \infty)$ and check a value in each.

$$f(-10) = \frac{13}{16},$$
$$f(-3) = -\frac{6}{5},$$
$$f(0) = 3,$$
$$f(5/2) = -\frac{13}{9},$$
$$f(4) = \frac{2}{3}.$$

Putting it all together, you should get something like this.



2. $g(x) = \dfrac{2x^2 + 2x - 24}{x^2 - x - 6}$ *(there's a cancellation – so what?)*

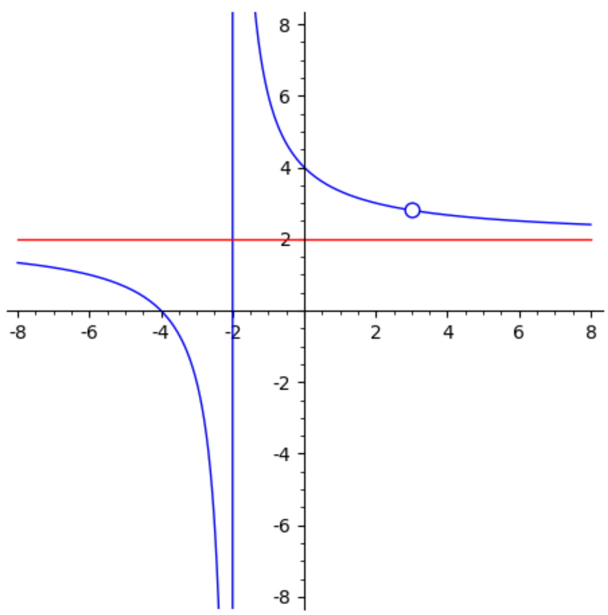
Same deal – take it one step at a time.

1. The numerator is $2(x+4)(x-3)$; the denominator is $(x-3)(x+2)$. Cancelling out the $x-3$'s, we get
$$g(x) = \frac{2(x+4)}{(x+2)}, \qquad x \neq 3,\, x \neq -2\ .$$
Important: even though $x-3$ doesn't appear in the denominator here, we're not allowed to plug it in to the function: it's not in the domain.
2. The zeroes are at $x = -4$ and $x = 3$.
3. The y-intercept is $g(0) = 4$.
4. There is only one vertical asymptote, at $x = -2$. Again, $x-3$ doesn't appear in the reduced form, so there's no asymptote!
5. The numerator and denominator have equal degree, so there is a horizontal asymptote. This time, it's $\frac{2}{1} = 2$.
6. Now we split the x-axis into the intervals $(-\infty, -4)$, $(-4, -2)$, $(-2, 2)$, $(2, 3)$, and $(3, \infty)$. This time, I'll only figure out if it's positive or negative. (Really, you can ignore the value $x = 3$ when you do this, and just use the simplified formula for the function – notice that it's positive on both sides of

$x = 3$. But there's no harm in doing extra work.)

$$g(-10) = \frac{2(-10+4)(-10-3)}{(-10+2)(-10-3)} = \frac{(-)(-)}{(-)(-)} = +$$
$$g(-3) = \frac{2(-3+4)(-3-3)}{(-3+2)(-3-3)} = \frac{(+)(-)}{(-)(-)} = -$$
$$g(0) = \frac{2(0+4)(0-3)}{(0+2)(0-3)} = \frac{(+)(-)}{(+)(-)} = +$$
$$g(3/2) = \frac{2(3/2+4)(3/2-3)}{(3/2+2)(3/2-3)} = \frac{(+)(-)}{(+)(-)} = +$$
$$g(4) = \frac{2(4+4)(4-3)}{(4+2)(4-3)} = \frac{(+)(+)}{(+)(+)} = +$$

The official graph looks like the following.



A word on $x = 3$. This is a "hole in the graph": the function isn't defined there, even though there is no asymptote. The usual practice is to draw a a small empty circle on the graph at this point to emphasize that $x = 3$ isn't in the domain.

3. $h(x) = \dfrac{x}{x^2 - 4}$

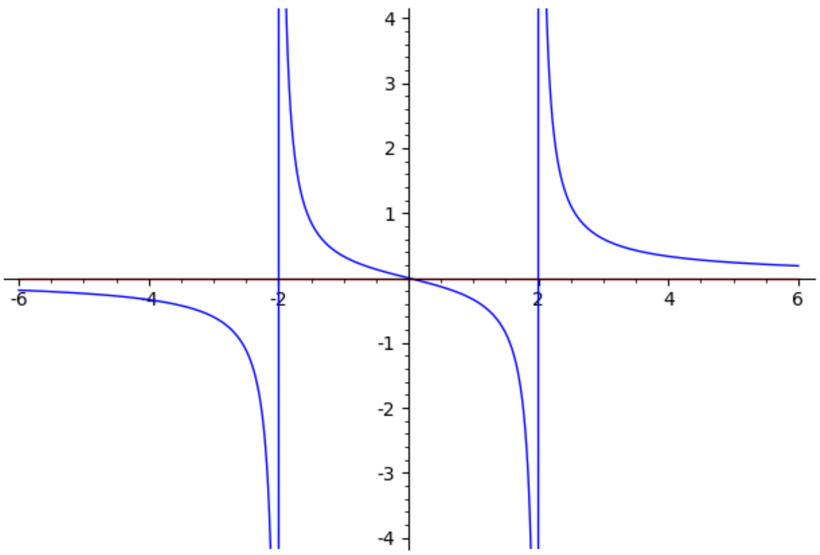
Here we go again.

1. The numerator is already factored. The denominator is $(x-2)(x+2)$. Nothing cancels.
2. The only zero is at $x = 0$.
3. The y-intercept is $h(0) = 0$.
4. There are vertical asymptotes at $x = -2$ and $x = 2$.
5. The degree of the denominator is bigger, so $y = 0$ is a horizontal asymptote.

6. Now we split the x-axis into the intervals $(-\infty, -2)$, $(-2, 0)$, $(0, 2)$, $(2, \infty)$. Again, figure out if it's positive or negative on each interval.

$$h(-3) = \frac{(-3)}{(-3+2)(-3-2)} = \frac{(-)}{(-)(-)} = -$$
$$h(-1) = \frac{(-1)}{(-1+2)(-1-2)} = \frac{(-)}{(+)(-)} = +$$
$$h(1) = \frac{(1)}{(1+2)(1-2)} = \frac{(+)}{(+)(-)} = -$$
$$h(3) = \frac{(3)}{(3+2)(3-2)} = \frac{(+)}{(+)(+)} = +$$

Now it's a matter of putting it all together. The trickiest part is the asymptotes. Here's what you should get:



We can see that the function appears to be approaching the horizontal asymptote.

4. $i(x) = \dfrac{x^2 + 3x + 2}{x - 1}$ *(there's a slant asymptote – can you figure out what it is?)*

1. The numerator is $(x+2)(x+1)$. The denominator is $x - 1$. Again, no cancellation.
2. The zeroes are at $x = -1$ and $x = -2$.
3. The y-intersect is -2.
4. The only vertical asymptote is at $x = 1$.
5. The degree of the asymptote is bigger, so no horizontal asymptote. There is, however, a slant asymptote, since the degree is only 1 bigger. To see what it is, use polynomial long division of $x^2 + 3x + 2$ by $x - 1$. You should get $x + 4$, with a remainder of 6. That means that

$$x^2 + 3x + 2 = (x + 4)(x - 1) + 6.$$

(You can check this! Just multiply out the right side and make sure they match up.) So our function is

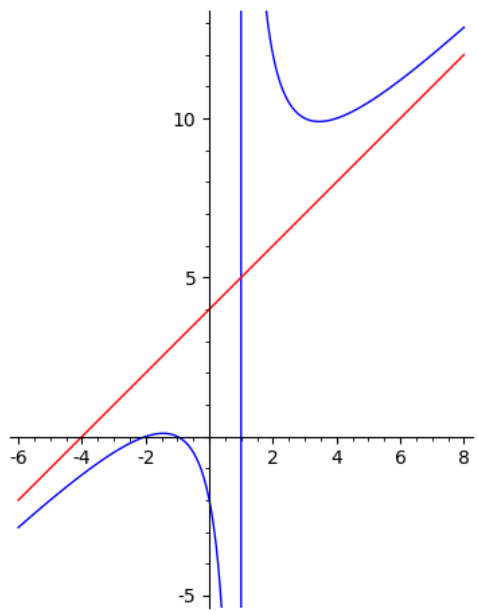
$$\frac{(x+4)(x-1)+6}{x-1} = x+4+\frac{6}{x-1}.$$

When x is big, the fractional part on the right is close to 0, so the function approaches $y = x + 4$. That's going to be our slant asymptote.

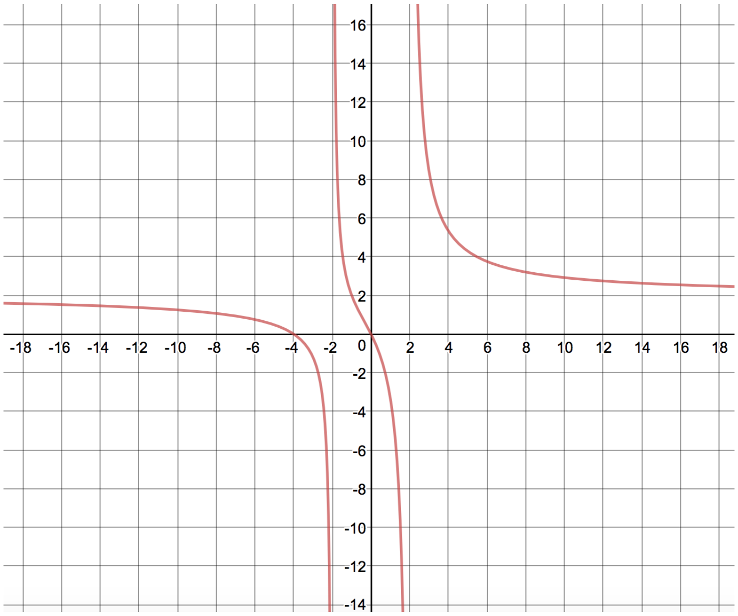
6. Now we split the x-axis into the intervals $(-\infty, -2)$, $(-2, -1)$, $(-1, 1)$, $(1, \infty)$. Again, figure out if it's positive or negative on each interval.

$$i(-3) = \frac{(-3+2)(-3+1)}{-3-1} = \frac{(-)(-)}{(-)} = -$$
$$i(-3/2) = \frac{(-3/2+2)(-3/2+1)}{-3/2-1} = \frac{(+)(-)}{(-)} = +$$
$$i(0) = \frac{(0+2)(0+1)}{0-1} = \frac{(+)(+)}{(-)} = -$$
$$i(2) = \frac{(2+2)(2+1)}{2-1} = \frac{(+)(+)}{(+)} = +$$

Putting everything together to draw a graph, we get:



5. *Given the graph below, find all asymptotes, intercepts, end behavior, domain, and range. Then use this to find a possible formula for the rational function.*

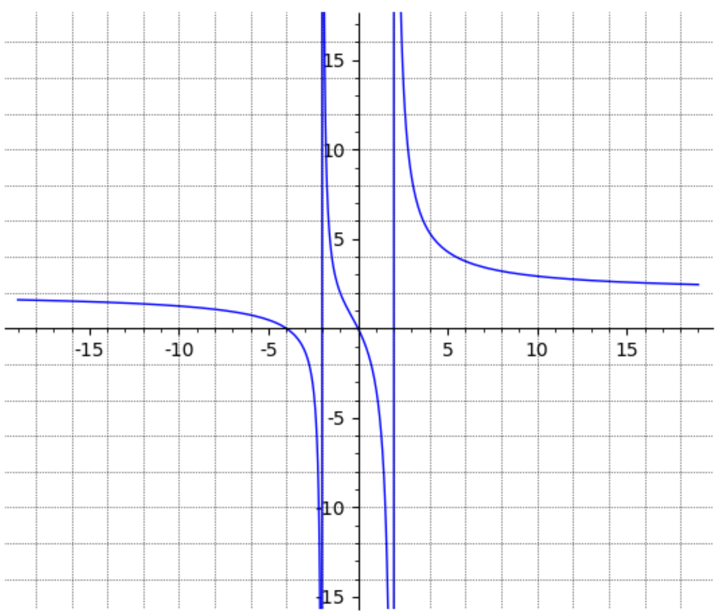


It looks to me like there are vertical asymptotes at $x = -2$ and $x = 2$. There is a horizontal asymptote at $y = 2$, and zeroes at $x = 0$ and $x = -4$. The domain is $x \neq -2$, $x \neq 2$, and the range is all real numbers.

The vertical asymptotes tell us the denominator is likely to be $(x-2)(x+2) = x^2-4$. The numerator looks like $x(x+4)$. There's a horizontal asymptote that isn't $y = 0$, so probably the degree of the numerator and the denominator are equal. However, we want the leading coefficient of the numerator to be twice that of the denominator, to get $y = 2$ as the horizontal asymptote. So, my first guess is this function:

$$j(x) = \frac{2x(x+4)}{(x+2)(x-2)} = \frac{2x^2+8x}{x^2-4}.$$

Here's a graph of that function, from the computer.



Looks pretty good to me.