Math 121 (Lesieutre); §5.1 & 5.2; September 8, 2017

1. *A* ***function*** *is a relation that maps each element, x, of the domain to exactly* **one** *element, y, of the range. You can check if a relation is a function using the vertical line test. We can also define* **composition of functions** *using:*

$$(f \circ g)(x) = f\,(g(x))\,.$$

2. *Given $f(x) = x^2 - 3x$ and $g(x) = x + 1$, find the following.*

a) *What is the domain of each function?*
In both cases, the domain is all real numbers: there are no values that we can't plug in.

b) *What are $f(g(x))$ and $g(f(x))$?*
$f(g(x))$ means that everywhere you see a x in $f(x)$, you should plug in the formula $g(x)$ instead. Then expand. Ditto for the other one, but with the roles reversed.

$$f(g(x)) = (x+1)^2 - 3(x+1) = x^2 - x - 2,$$
$$g(f(x)) = (x^2 - 3x) + 1 = x^2 - 3x + 1.$$

c) *What is the domain of your answers in part (b)?*
Again, both of these are defined for all real numbers.

d) *What is $f(g(-1))$?*
This means to actually plug in the number -1.

$$g(-1) = (-1) + 1 = 0,$$
$$f(g(-1)) = f(0) = 0.$$

3. *Sometimes you have to think a little harder about the domain of a composed function: you need to restrict to x's for which the output of the inner function is a valid input to the second function. For the pairs below, find the composition $(f \circ g)(x)$, and determine its domain.*

a) $f(x) = \sqrt{x+1}$, $g(x) = 3x$.
Here $f(g(x)) = \sqrt{3x+1}$. This is defined as long as $3x + 1 \geq 0$, which is equivalent to saying $x \geq -\frac{1}{3}$.

b) $f(x) = \dfrac{x}{x+3}, \quad g(x) = \dfrac{2}{x}.$

Here

$$f(g(x)) = \frac{\frac{2}{x}}{\frac{2}{x}+3} = \frac{\frac{2}{x}}{\frac{2+3x}{x}} = \frac{2}{x} \cdot \frac{x}{2+3x} = \frac{2}{3x+2}.$$

How about the domain? It's all x for which x is in the domain of g (that rules out $x = 0$) and we $g(x)$ is in the domain of $f(x)$. The domain of $f(x)$ is everything except $x = -3$, so the problem is if $g(x) = -3$. That means $2/x = -3$, so $x = -2/3$. So the domain is

$$\left\{x | x \neq 0,\, x \neq -\frac{2}{3}\right\}.$$

4. *Now let's* de*compose functions. For the H's given, find f and g such that* $f \circ g = H$.

$$H(x) = (2x+3)^7, \qquad H(x) = \sqrt{1-x^2}, \qquad H(x) = \frac{1}{3x^3-1}.$$

This is a very important one to remember for Math 180 and 181! You want to untangle this by thinking to yourself: when I calculate this function, what do I do first, and what do I do second? For example, to get $(2x+3)^7$, you would first compute $2x+3$, and then compute the seventh power of that. So the "inner" function g is $2x+3$, and the outer one is x^7.

$$f(x) = x^7 \qquad f(x) = \sqrt{x} \qquad f(x) = \frac{1}{x},$$

$$g(x) = 2x+3 \qquad g(x) = 1-x^2 \qquad g(x) = 3x^3-1.$$

5. *What is an inverse function?*

Suppose that $f(x)$ is a one-to-one function: that means you never have two different inputs that give the same output (you may have learned the "horizontal line test").

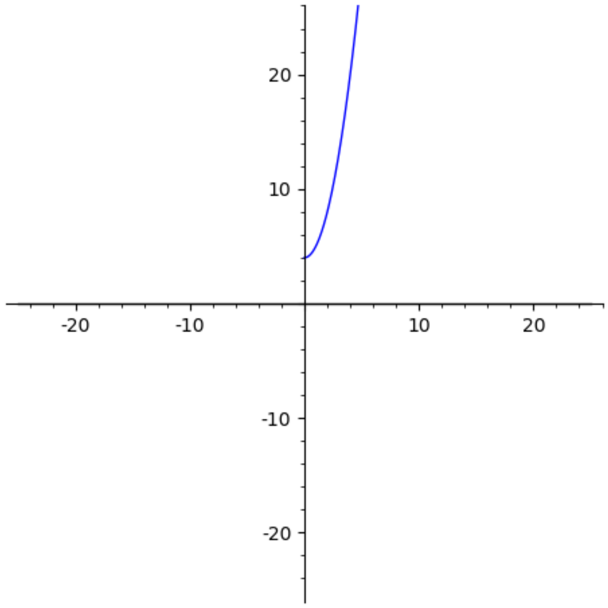
An inverse function $f^{-1}(x)$ is a function for which $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$. Think: whatever f does to a number x, f^{-1} *undoes* it and gives back the original number.

6. *Let* $f(x) = x^2 + 4$ *with domain* $x \geq 0$. *Find* f^{-1}. *(Why was the domain restricted?) Sketch a graph of both.*

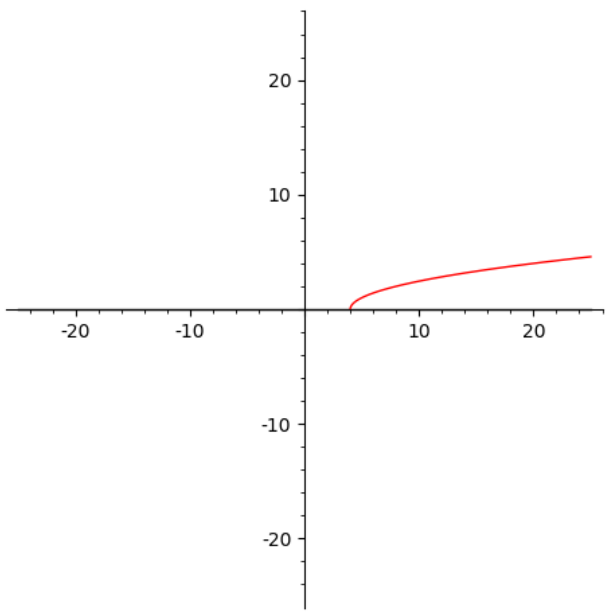
To find an inverse function: set y equal to your original function, swap x and y, and then solve for y. In this case, we start with $y = x^2 + 4$, swap it to get $x = y^2 + 4$, and then solve: $y^2 = x - 4$, $y = \sqrt{x-4}$. The inverse function is $y = \sqrt{x-4}$, with domain $x \geq 4$.

The reason we had to restrict the domain is that otherwise our original function isn't one-to-one. We have $f(-2) = 8$ and $f(2) = 8$, for example.

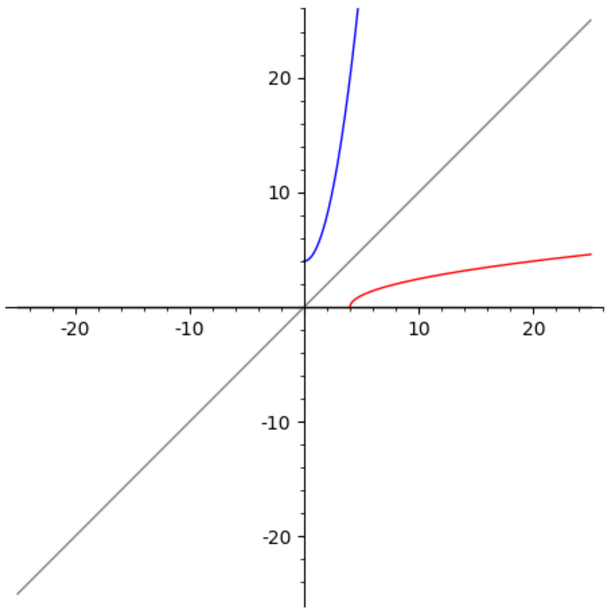
Here's the graph of the function:



And the inverse:



You probably notice something odd: the graph of the inverse function is the same of the graph of the original function, but reflected over the line $y = x$.



In fact, that's how it always works. To get the graph of the inverse, flip your graph about the line $y = x$.

7. *Show that the following are inverse functions by showing* $f(g(x)) = x = g(f(x))$.

$$f(x) = x^3 - 8, \qquad g(x) = \sqrt[3]{x+8}$$

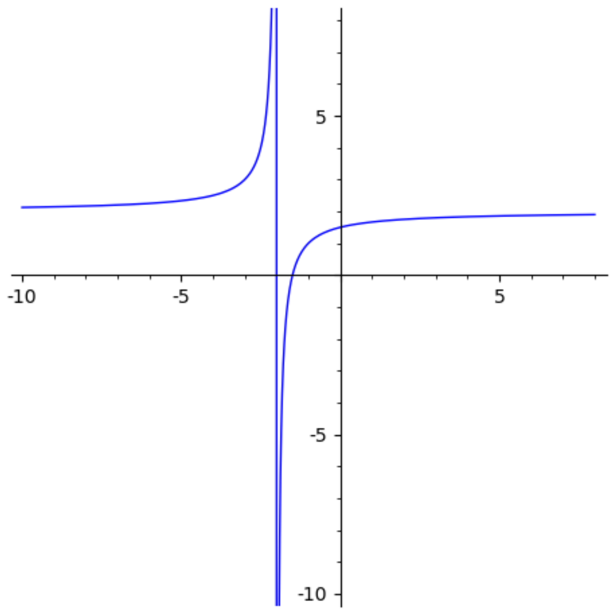
$$\begin{aligned} f(g(x)) &= (\sqrt[3]{x+8})^3 - 8 = (x+8) - 8 = x, \\ g(f(x)) &= \sqrt[3]{(x^3-8)+8} = \sqrt[3]{x^3} = x. \end{aligned}$$

That's what we wanted.

8. *Given the rational function* $f(x) = \dfrac{2x+3}{x+2}$, *answer the following.*

a) *What is the domain of* f*? Sketch a graph.*

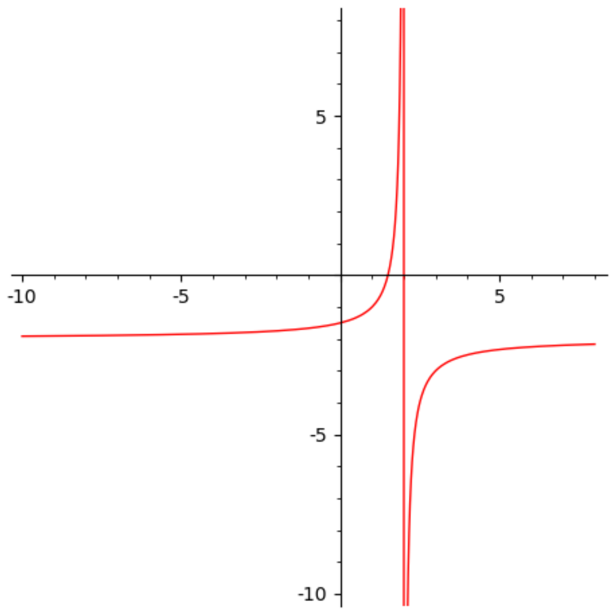
The domain is $x \neq -2$, which is what makes the denominator 0. To make a graph, you should go through the six steps we talked about last time. I'm going to cheat and just use a computer:



b) *Find* f^{-1} *and sketch the graph.*
This time we take $x = \frac{2y+3}{y+2}$ and solve for y:

$$\begin{aligned}(y+2)x &= 2y+3\\ xy+2x &= 2y+3\\ xy-2y &= 3-2x\\ (x-2)y &= 3-2x\\ y &= \frac{3-2x}{x-2}\end{aligned}$$

That's the inverse. Here's the plot:



Once again the graphs are the same, except flipped over $x = y$: here they are together, which makes this a little easier to see: