Math 210 (Lesieutre)
11.3: Dot products
January 13, 2017

Problem 1. *For each of the following, try to estimate the dot product without making any calculations. Then check your answer using the formula.*

a) $\langle 3, 4\rangle \cdot \langle 4, 3\rangle$

These both have length 5, and point in roughly the same direction. That means $\cos\theta$ is going to be positive and more or less close to 1. So we expect the dot product to be a little less than 25.

The correct value is $(3)(4) + (4)(3) = 24$.

b) $\langle 3, 1\rangle \cdot \langle -1, 4\rangle$

If you sketch these, you'll see that they're almost perpendicular, but not quite. The dot product will be small, at least compared to the lengths of the vectors. Positive or negative might be hard to tell unless you draw a good picture, but it should be slightly positive.

The actual value is 1.

c) $\langle 1, 0, 0\rangle \cdot \langle 0, 1, -1\rangle$

The first vector points along the x-axis, while the second lies in the yz-plane . The two vectors are orthogonal, and the dot product will be exactly 0.

The formula confirms this.

d) $\langle 1, 2, 3\rangle \cdot \langle -2, -2, -2\rangle$

Both vectors have length somewhere between 3 and 4. They point in basically opposite directions, so $\cos\theta$ is going to be fairly close to -1. So the answer should be a negative number in the ballpark of -10.

Multiplying it all out, you'll find the actual value is -12.

Problem 2. *Compute the angle between the two vectors in 1(a). (Your answer might be in terms of an* $\cos^{-1}$*).*

We know that $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}|\,|\mathbf{v}| \cos\theta$, so

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|} = \frac{24}{25}.$$

That means $\theta = \cos^{-1}(24/25)$, which is about 16°.

Problem 3. *Suppose that* $\mathbf{v} = \langle a, b\rangle$ *is a vector. What is* $\mathbf{v} \cdot \mathbf{v}$*?*

It's $\langle a, b\rangle \cdot \langle a, b\rangle = a^2 + b^2 = |\mathbf{v}|^2$. This works for 3d vectors too!

Problem 4. *Sketch the vectors* $\mathbf{u} = \langle 2, 1\rangle$ *and* $\mathbf{v} = \langle 1, 1\rangle$*, and draw the vector* $\operatorname{proj}_{\mathbf{v}} \mathbf{u}$*. What is* $\operatorname{scal}_{\mathbf{v}} \mathbf{u}$*?*

(Sorry, I'm omitting the drawing.)

The formula for projection gives us

$$\operatorname{proj}_{\mathbf{v}} \mathbf{u} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v} = \left(\frac{3}{2}\right)\langle 1, 1\rangle = \left\langle \frac{3}{2}, \frac{3}{2}\right\rangle.$$

This is a vector in the same direction as $\mathbf{v}$, and represents the part of $\mathbf{u}$ that's in the direction of $\mathbf{v}$.

For the scalar component, we use

$$\operatorname{scal}_{\mathbf{v}} \mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|} = \frac{3}{\sqrt{2}} = \frac{3\sqrt{2}}{2}.$$

This is the length of $\operatorname{proj}_{\mathbf{v}} \mathbf{u}$.

Problem 5. *A force* $\mathbf{F} = 2, 1, 1$ *(Newtons) pushes an object from* $(1, 0, 0)$ *to* $(3, 0, 0)$ *(meters). Calculate the work done (Joules).*

It's just $W = \mathbf{F}\cdot\mathbf{d} = \langle 2, 1, 1\rangle \cdot \langle 2, 0, 0\rangle = 4$.

Problem 6. *A* 20 *pound block sits on a plane with slope* 30 *degrees. Compute the components of the gravitation force that are parallel and perpendicular to the plane.*

The force is $\mathbf{F} = \langle 0, -20\rangle$. The vector "down" the slope is $\mathbf{v} = \left\langle \sqrt{3}2, -\frac{1}{2}\right\rangle$. The vector into the plane is $\mathbf{u} = \left\langle \frac{1}{2}, -\frac{\sqrt{3}}{2}\right\rangle$.

We find

$$\operatorname{proj}_{\mathbf{v}} \mathbf{F} = \left(\frac{\mathbf{F}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v} = \left\langle 5\sqrt{3}, -5\right\rangle$$

and

$$\operatorname{proj}_{\mathbf{u}} \mathbf{F} = \left(\frac{\mathbf{F}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u} = \left\langle -5\sqrt{3}, -15\right\rangle.$$

The one into the plane is usually called the normal component and written $\mathbf{N}$.

Does the object slide down the plane? It's a question of whether the frictional force is strong enough to overcome the force $\operatorname{proj}_{\mathbf{v}} \mathbf{F}$. This depends on the coefficient of friction, which takes us a little further into physics than we're really going to go...

Problem 7. *a) Describe the set of all vectors* $\mathbf{v}$ *for which* $\mathbf{v}\cdot\langle 1, 1, 1\rangle = 0$.

This is all vectors that are perpendicular to $\langle 1, 1, 1\rangle$*. These form the plane whose normal vector is* $\langle 1, 1, 1\rangle$.

b) For which vectors is $\text{proj}_{\langle 1,0,0 \rangle} = 5$*?*
This works for any vector $\langle 5, a, b \rangle$*; the set of such vectors is the plane* $x = 5$*.*