Math 210 (Lesieutre)
11.7: Motion in space
January 25, 2017

Problem 1. *Consider the path* $\mathbf{r}(t) = \langle \cos t, \sin t, t^2 \rangle$. *Compute the velocity, speed, and acceleration, all as functions of* t.

First let's find the velocity and acceleration, which are just a matter of taking derivatives. We get

$$\begin{aligned} \mathbf{r}(t) &= \left\langle \cos t, \sin t, t^2 \right\rangle, \\ \mathbf{v}(t) &= \langle -\sin t, \cos t, 2t \rangle, \\ \mathbf{a}(t) &= \langle -\cos t, -\sin t, 2 \rangle. \end{aligned}$$

The speed should be a scalar function, and to get it we take the length of velocity, as a function of time:

$$|\mathbf{v}(t)| = \sqrt{(-\sin t)^2 + (\cos t)^2 + (2t)^2} = \sqrt{1 + 4t^2}.$$

This seems plausible, I think. When t is big, it's moving upwards faster and faster, which is reflected in the speed increasing as t increases.

Problem 2. *A particle moves in a circular pattern, given by* $\mathbf{r}(t) = \langle 2\cos t, 2\sin t \rangle$.

a) *Compute the velocity, acceleration, and speed, as functions of* t.

Taking derivatives, we get

$$\begin{aligned} \mathbf{r}(t) &= \langle 2\cos t, 2\sin t \rangle, \\ \mathbf{v}(t) &= \langle -2\sin t, 2\cos t \rangle, \\ \mathbf{a}(t) &= \langle -2\cos t, -2\sin t \rangle. \end{aligned}$$

The speed is

$$|\mathbf{v}(t)| = \sqrt{(-2\sin t)^2 + (2\cos t)^2} = \sqrt{4} = 2,$$

which doesn't depend on t. This makes sense, since it's moving at a constant rate. Notice that the velocity vector $\mathbf{v}(t)$ *does* depend on t: this reflects the fact that the direction is changing.

b) *Sketch the path of the particle. For* $t = \pi/4$, *draw the vectors* $\mathbf{r}(t)$, $\mathbf{v}(t)$, *and* $\mathbf{a}(t)$. *Do these seem to make physical sense?*

The vectors are $\mathbf{r}(t) = \left\langle \sqrt{2}, \sqrt{2} \right\rangle$, $\mathbf{v}(t) = \left\langle -\sqrt{2}, \sqrt{2} \right\rangle$, and $\mathbf{a}(t) = \left\langle -\sqrt{2}, \sqrt{2} \right\rangle$.

The velocity vector is tangent to the circle at the point, which makes sense: velocity is supposed to be the tangent vector. The acceleration points in to the circle, which may seem a little weird at first, but this is how things work for circular motion. This is consistent with Newton's law $F = ma$: because gravitation force pulls the particle inward, the acceleration should also be inward.

Problem 3. *Suppose that a particle moves in a straight-line path* $\mathbf{r}(t) = \mathbf{r}_0 + \mathbf{v}t$. *What are the velocity and acceleration?*

The velocity will just be the vector $\mathbf{v}$, while the acceleration is the zero vector $\mathbf{0}$. (Because the particle is moving at constant velocity, there is no acceleration.)

Problem 4. *A particle has acceleration* $\mathbf{a}(t) = \langle 2t, -1\rangle$. *Suppose that at time* 0*, it has position* $\mathbf{r}(0) = \langle 2, 3\rangle$ *and velocity* $\mathbf{v}(t) = \langle 0, 1\rangle$. *Find* $\mathbf{r}(t)$.

We know that

$$\mathbf{v}(t) = \int \mathbf{a}(t)\, dt = \int \langle 2t, -1\rangle\; dt = \left\langle t^2, -t\right\rangle + \mathbf{C}_1$$

Since $\mathbf{v}(0) = \langle 0, 1\rangle$, we get $\langle 0, 0\rangle + \mathbf{C}_1 = \langle 0, 1\rangle$, so $\mathbf{C}_1 = \langle 0, 1\rangle$. That means

$$\mathbf{v}(t) = \left\langle t^2, -t\right\rangle + \langle 0, 1\rangle = \left\langle t^2, 1 - t\right\rangle.$$

Then

$$\mathbf{r}(t) = \int \mathbf{v}(t)\, dt = \int \left\langle t^2, 1 - t\right\rangle\; dt = \left\langle \frac{t^3}{3}, t - \frac{t^2}{2}\right\rangle + \mathbf{C}_2$$

Plugging in $t = 0$, we get

$$\langle 0, 0\rangle + \mathbf{C}_2 = \mathbf{r}(0) = \langle 2, 3\rangle,$$

which means that $\mathbf{C}_2 = \langle 2, 3\rangle$. So at last we have

$$\mathbf{r}(t) = \left\langle \frac{t^3}{3}, t - \frac{t^2}{2}\right\rangle + \langle 2, 3\rangle = \left\langle \frac{t^3}{3} + 2, t - \frac{t^2}{2} + 3\right\rangle.$$

Problem 5. *A batter hits a baseball with initial velocity* $\langle 100, 100, 100\rangle$ *(a pop-up down the right field line; let's say the units are ft/sec).*

a) *What is* $\mathbf{a}(t)$*? Assume the only force acting on the ball is gravity.*
It's a constant downward force, $\langle 0, 0, -32\rangle$. The 32 here is gravitation acceleration.

b) *Solve for* $\mathbf{v}(t)$ *and* $\mathbf{r}(t)$.
Well,

$$\mathbf{v}(t) = \int \mathbf{a}(t)\, dt = \langle 0, 0, -32t\rangle + \mathbf{C}_1.$$

Since $\mathbf{v}(0) = \langle 100, 100, 100\rangle$, plugging in $t = 0$ we find that $\mathbf{C}_1 = \langle 100, 100, 100\rangle$. This means that

$$\mathbf{v}(t) = \langle 0, 0, -32t\rangle + \langle 100, 100, 100\rangle = \langle 100, 100, 100 - 32t\rangle.$$

Then

$$\mathbf{r}(t) = \int \mathbf{v}(t)\, dt = \left\langle 100t, 100t, 100t - 16t^2\right\rangle + \mathbf{C}_2.$$

Since $\mathbf{r}(0) = \mathbf{0}$, the constant vector $\mathbf{C}_2$ is 0 too. Thus

$$\mathbf{r}(t) = \left\langle 100t, 100t, 100t - 16t^2\right\rangle.$$

c) *At what time t does the ball hit the ground?*

This will happen when $100t - 16t^2 = 0$, so that $t = 100/16 = 6.25$ seconds.

Problem 6. a) *What is*

$$\frac{d}{dt}\left|\mathbf{x}(t)\right|^2,$$

in terms of $\mathbf{x}(t)$ *and its derivatives?*

This is

$$\frac{d}{dt}\mathbf{x}(t)\cdot\mathbf{x}(t) = \mathbf{x}(t)\cdot\mathbf{x}'(t) + \mathbf{x}'(t)\cdot\mathbf{x}(t) = 2\mathbf{x}(t)\cdot\mathbf{x}'(t).$$

b) *Suppose that* $\mathbf{x}(t)$ *has constant length. What does this tell you about the relationship between* $\mathbf{x}(t)$ *and* $\mathbf{x}'(t)$*?*

It tells you that they are orthogonal for all values of t!