Math 210 (Lesieutre)
11.8: Length of curves
January 27, 2017

Problem 1. *Let R be a circle centered at 0 with radius 5. Set up a parametrization $\mathbf{r}(t)$ for the circle, and use it to compute the circumference.*

This one is pretty painless. Use

$$\mathbf{r}(t) = \langle 5\cos t, 5\sin t\rangle$$

with $0 \le t \le 2\pi$. Then $\mathbf{r}'(t) = \langle -5\sin t, 5\cos t\rangle$, which gives us

$$|\mathbf{r}'(t)| = \sqrt{(-5\sin t)^2 + (5\cos t)^2} = \sqrt{25} = 5.$$

This doesn't depend on t! So the arc length is

$$\int_0^{2\pi} 5\,dt = 10\pi.$$

This matches up with the $2\pi R$ you probably learned before you took calculus.

Problem 2. *Consider the curve $\mathbf{r}(t) = \left\langle \cos t, \sin t, \frac{2}{3}t^{3/2}\right\rangle$. Compute the length of this curve between $t = 0$ and $t = 10$.*

We have

$$\mathbf{r}'(t) = \left\langle -\sin t, \cos t, t^{1/2}\right\rangle,$$

which gives

$$|\mathbf{r}'(t)| = \sqrt{(-\sin t)^2 + (\cos t)^2 + t} = \sqrt{1+t}.$$

The formula for arc length then tells us that

$$L = \int_0^{10} \sqrt{1+t}\,dt = \left(\frac{2}{3}(1+t)^{3/2}\right)\Big|_0^{10} = \frac{2}{3}11^{3/2} - \frac{2}{3} = \frac{2}{3}(11\sqrt{11} - 1).$$

Sanity check? That's about 23.65, which seems in the right ballpark at least.

Problem 3. *Consider a cardioid $r = 1 + \cos\theta$. Set up the integral for the arc length, and evaluate it if you can.*

We have $f(\theta) = 1 + \cos\theta$ and so $f'(\theta) = -\sin\theta$. The formula we just saw gives

$$\begin{aligned} L &= \int_\alpha^\beta \sqrt{f(\theta)^2 + f'(\theta)^2}\, d\theta \\ &= \int_0^{2\pi} \sqrt{(1+\cos\theta)^2 + (-\sin\theta)^2}\, d\theta \\ &= \int_0^{2\pi} \sqrt{1 + 2\cos\theta + \cos^2\theta + \sin^2\theta}\, d\theta \\ &= \int_0^{2\pi} \sqrt{2 + 2\cos\theta}\, d\theta = 2\int_0^{\pi} \sqrt{2+2\cos\theta}\, d\theta \\ &= \int_0^{2\pi} \sqrt{4\cos^2(\theta/2)}\, d\theta. \end{aligned}$$

There is a subtle point here. It's pretty tempting to just write $\sqrt{4\cos^2(\theta/2)} = 2\cos(\theta/2)$. But here's the catch: it's really $|2\cos(\theta/2)|$, which isn't the same thing if $\cos(\theta/2)$ is negative. So we can't make this simplification. However, notice that the cardioid is symmetrical: the length from 0 to π is equal to the length from π to 2π, so it is true that

$$\int_0^{2\pi} \sqrt{4\cos^2(\theta/2)}\, d\theta = 2\int_0^{\pi} \sqrt{4\cos^2(\theta/2)}\, d\theta.$$

Now, when $0 \le \theta \le \pi$, we have $0 \le \theta/2 \le \pi/2$, and so $\cos(\theta/2)$ is guaranteed to be positive. That means for θ in this range, it's actually true that $\sqrt{4\cos^2(\theta/2)} = 2\cos(\theta/2)$. So our integral becomes

$$4\int_0^{\pi} \cos(\theta/2) d\theta = 8\sin(\theta/2)|_0^{\pi} = 8.$$

That was a little painful, but I think the result is interesting: the length of the cardioid is 8. No square roots, no π, just 8.

Problem 4. *The circle in the first problem was not parametrized by arc length. Give another parametrization in which it is.*

The issue is that the particle is moving too fast: it's going at a constant speed of 5. If we use $\mathbf{r}(t) = \langle 5\cos(t/5), 5\sin(t/5)\rangle$ instead, it will slow down, and now is parametrized by arc length. Then $\mathbf{r}'(t) = \langle -\sin(t/5), \cos(t/5)\rangle$, which gives us

$$|\mathbf{r}'(t)| = \sqrt{(-\sin(t/5))^2 + (\cos(t/5))^2} = \sqrt{1} = 1,$$

which is what it means to be parametrized by arc length.

Problem 5. *A batter hits a baseball with initial velocity* $\langle 100, 100, 100\rangle$ *(a pop-up down the right field line; let's say the units are ft/sec).*

a) *What is* $\mathbf{a}(t)$*? Assume the only force acting on the ball is gravity.*
It's a constant downward force, $\langle 0, 0, -32\rangle$. The 32 here is gravitational acceleration.

b) *Solve for* $\mathbf{v}(t)$ *and* $\mathbf{r}(t)$*.*
Well,

$$\mathbf{v}(t) = \int \mathbf{a}(t)\,dt = \langle 0, 0, -32t\rangle + \mathbf{C}_1.$$

Since $\mathbf{v}(0) = \langle 100, 100, 100\rangle$, plugging in $t = 0$ we find that $\mathbf{C}_1 = \langle 100, 100, 100\rangle$. This means that

$$\mathbf{v}(t) = \langle 0, 0, -32t\rangle + \langle 100, 100, 100\rangle = \langle 100, 100, 100 - 32t\rangle.$$

Then

$$\mathbf{r}(t) = \int \mathbf{v}(t)\,dt = \left\langle 100t, 100t, 100t - 16t^2\right\rangle + \mathbf{C}_2.$$

Since $\mathbf{r}(0) = \mathbf{0}$, the constant vector $\mathbf{C}_2$ is 0 too. Thus

$$\mathbf{r}(t) = \left\langle 100t, 100t, 100t - 16t^2\right\rangle.$$

c) *At what time* t *does the ball hit the ground?*
This will happen when $100t - 16t^2 = 0$, so that $t = 100/16 = 6.25$ seconds.

d) *Set up the integral for the length of the path of the ball between the moment it left the bat and the moment it hit the ground.*
We have $\mathbf{r}'(t) = \langle 100, 100, 100 - 32t\rangle$, and so

$$L = \int_0^{6.25} |\mathbf{r}'(t)|\,dt = \int_0^{6.25} \sqrt{100^2 + 100^2 + (100 - 32t)^2}\,dt = \int_0^{6.25} \sqrt{20000 + (100 - 32t)^2}\,dt$$
$$= \frac{625}{2}\left(\sqrt{3} + 2\,\mathrm{arcsinh}\left(\frac{\sqrt{2}}{2}\right)\right) \approx 952.815.$$

(I'd say you don't need to worry about how to actually do that integral.)