Math 210 (Lesieutre)
12.6: Directional derivatives and the gradient
February 10, 2017

Problem 1. *Suppose that $f(x,y) = xy^2$, and $x(s,t) = 2s + t$ and $y(s,t) = s\cos t$. Compute the partial derivative $\frac{\partial f}{\partial s}$.*

This is a little more painful. Again we have to use the chain rule.

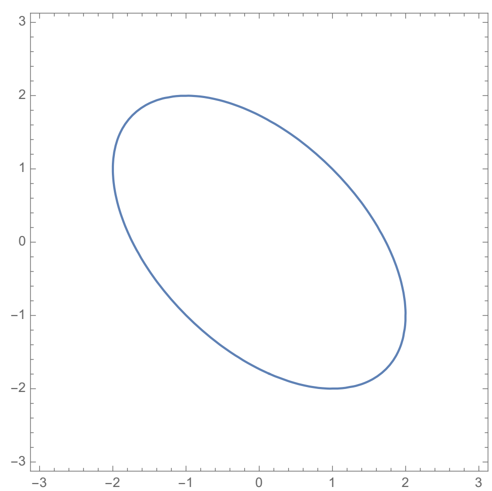
$$\begin{aligned}\frac{\partial f}{\partial s} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} \\ &= (y^2)(2) + (2xy)(\cos t) = 2(s\cos t)^2 + (2)(2s+t)(s\cos t)(\cos t) \\ &= 2s^2\cos^2 t + (4s^2 + 2st)\cos^2 t = 6s^2\cos^2 t + 2st\cos^2 t.\end{aligned}$$

Problem 2. *Consider the ellipse defined by $F(x,y) = 0$, where $F(x,y) = x^2 + xy + y^2 - 1$. Compute $\frac{dy}{dx}$.*

The formula for implicit differentiation gives

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{2x+y}{2y+x}.$$

Let's make sure we actually understand what this is saying. Here's the ellipse:



(It's tilted because of the xy-term; you probably haven't plotted one like that, and it's tough until you've taken Math 310.)

One point on the ellipse is $(x,y) = (1,1)$. Our formula is telling us that $\frac{dy}{dx}$, which is the slope of the tanegnt line, is given by $-\frac{2x+y}{2y+x} = -\frac{3}{3} = -1$ at this point. That appears to match up with the figure. So we were able to find the slope of the tangent line, even though we don't actually have a formula for y as a function of x.

Problem 3. *Consider the function* $f(x,y) = 10 - x^2 - 4y^2$.

a) *Compute the gradient* $\nabla f(x,y)$.
The gradient is given by $\langle f_x, f_y \rangle = \langle -2x, -8y \rangle$.

b) *Find the derivative in the direction of the vector* $\mathbf{v} = \langle 1, 1 \rangle$ *at the point* $(1,1)$. *(Watch out! This isn't a unit vector.)*

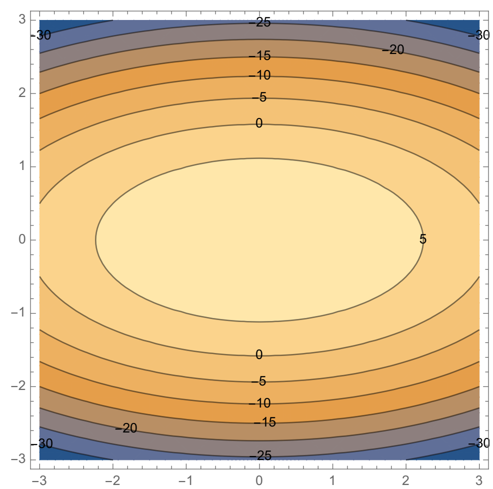
First we need to know a unit vector in the direction of $\mathbf{v}$. That's given by $\mathbf{u} = \left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle$. Now the directional derivative is

$$D_{\mathbf{u}} f(a,b) = \nabla f(a,b) \cdot \mathbf{u} = \langle -2, -8 \rangle \cdot \left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle$$
$$= -\sqrt{2} - 4\sqrt{2} = -5\sqrt{2}.$$

c) *Find the directional derivative in the direction of the vector* $\mathbf{u} = \langle 0, -1 \rangle$ *at* $(1,1)$.
Same as strategy as above:

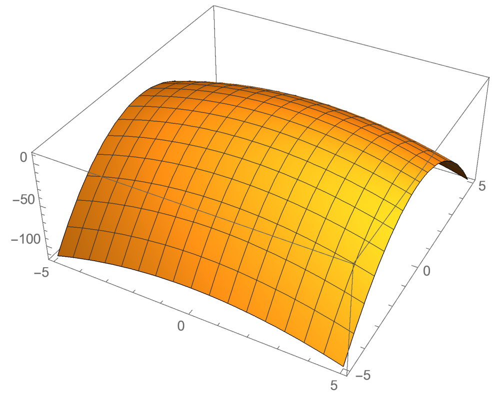
$$D_{\mathbf{u}} f(a,b) = \nabla f(a,b) \cdot \mathbf{u} = \langle -2, -8 \rangle \cdot \langle 0, -1 \rangle = 8.$$

d) *Sketch some level curves of* $f(x,y)$, *including the level curve with* $z = 5$.
The level curves look like $x^2 + 4y^2 = C$, which are ellipses. Here are a few.



$z = 5$ is the innermost one that's marked. Notice that the point $(1,1)$ is contained in this level curve.

Here's a graph of the whole surface, for what it's worth:



e) *Find the unit vectors in the directions of steepest ascent and descent at the point* $(1,1)$. *Do your answers make sense?*

The gradient is given by $\langle -2,-8\rangle$, which means steepest ascent is in the direction of $\langle -2,-8\rangle$. A unit vector in this direction is $\left\langle -2/\sqrt{68}, -8/\sqrt{68}\right\rangle$ (a mess, sorry).

Looking at this on the plot, this vector points "inwards" on the ellipse. Makes sense: the whole surface is shaped like a "hill", and the inwards direction is uphill.

Steepest descent is in the opposite direction, which is $\left\langle 2/\sqrt{68}, 8/\sqrt{68}\right\rangle$. This is "downhill", the direction a ball would roll if placed on the graph.

f) *Find the directional derivative in the direction of steepest ascent. Is this steeper than the answers you got for directional derivatives earlier in the problem?*

The directional derivative is given by

$$D_{\mathbf{u}}f(a,b) = \nabla f(a,b)\cdot\mathbf{u} = \langle -2,-8\rangle\cdot\left\langle -2/\sqrt{68}, -8/\sqrt{68}\right\rangle = 68/\sqrt{68} = \sqrt{68}.$$

This is slightly more than the 8 that we got in part (c), so it's plausible that this is indeed the direction of steepest ascent.

g) *Find a direction that is tangent to the level curve* $z=5$ *at the point* $(1,1)$. *What is the directional derivative in this direction?*

We want a direction that's perpendicular to the gradient. One option is $\left\langle 8/\sqrt{68}, -2/\sqrt{68}\right\rangle$, which will work because the dot product is 0. This points to the right and slightly down, which looks plausible for a tangent direction to the level curve at $(1,1)$, based on the picture.

The directional derivative is 0, because it's tangent to a level curve: the function doesn't change in this direction. Taking the dot product with the gradient confirms this.

Problem 4. *Suppose that a function has gradient* $\nabla f(0,0) = (1,1)$.

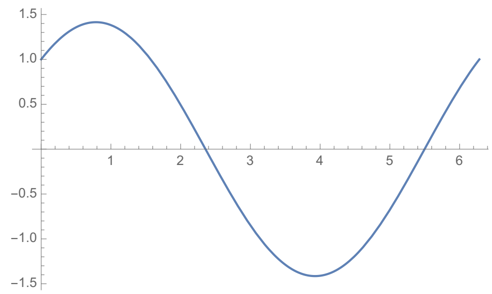
a) *What is the directional derivative of this function in a direction with angle* θ*?*

The unit vector we want is $\langle \cos\theta, \sin\theta\rangle$, and so the directional derivative is

$$\nabla f(0,0)\cdot\mathbf{u} = \cos\theta + \sin\theta.$$

b) *Plot the directional derivative for* $0 \leq \theta \leq 2\pi$. *For what* θ *is it maximized? Zero?*

Here's a plot:



It's maximized at $\theta = \pi/4$, which is the direction parallel to ∇f. It's 0 at $3\pi/4$ and $7\pi/4$, which is orthogonal to the gradient.