Math 210 (Lesieutre)
Exam 1 review
February 15, 2017

Problem 1. *Find the parametrization for a line which...*

a) ... *has* $\mathbf{r}(0) = (1, 2, 3)$ *and* $\mathbf{r}(1) = (1, -2, 1)$.
For the point $\mathbf{r}_0$, use $1, 2, 3)$. For the direction, use $(1, -2, 1) - (1, 2, 3) = (0, -4, -2)$. So the equation is

$$\mathbf{r}(t) = \mathbf{r}_0 + \mathbf{v}t = \langle 1, 2, 3 \rangle + t \langle 0, -4, 2 \rangle = \langle 1, 2 - 4t, 3 + 2t \rangle .$$

b) ... *is normal to the plane* $3x - 2y + z = 0$ *and passes through the origin.*
We want the line to be in the direction of the normal vector, so $\mathbf{v} = \langle 3, -2, 1 \rangle$.
The point is $\mathbf{r}_0 = \langle 0, 0, 0 \rangle$, and so:

$$\mathbf{r}(t) = \langle 0, 0, 0 \rangle + \langle 3, -2, 1 \rangle t = \langle 3t, -2t, t \rangle .$$

c) ... *is the intersection of the planes* $x + y + z = 3$ *and* $x - y + 2z = 1$.
This time the direction should be the cross product of the normal vectors, which is $\langle 1, 1, 1 \rangle \times \langle 1, -1, 2 \rangle$. This comes out to

$$\langle 1, 1, 1 \rangle \times \langle 1, -1, 2 \rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 1 & -1 & 2 \end{vmatrix} = \langle 3, -1, -2 \rangle .$$

We also need to find a point on the line. There are many, so to narrow down our search and make it so there's only one answer (which is then easy to find), let's try to find a point with $z = 0$. Then we need $x + y = 3$ and $x - y = 1$. The solution is $x = 2$, $y = 1$, and so our point is $\langle 2, 1, 0 \rangle$. That means the line in question is given by

$$\mathbf{r}(t) = \langle 2, 4, 8 \rangle + \langle 1, 4, 12 \rangle t = \langle 2 + t, 4 + 4t, 8 + 12t \rangle .$$

d) ... *is tangent to the curve* $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$ *at* $t = 2$.
The direction is given by the derivative, which is $\mathbf{r}'(t) = \langle 1, 2t, 3t^2 \rangle$. We want to plug in $t = 2$ to get the direction vector at that time, and so the direction of the tangent line is $\mathbf{v} = \langle 1, 4, 12 \rangle$.
What point does the tangent line need to go through? It's $\mathbf{r}(2)$ itself, which is $\langle 2, 4, 8 \rangle$. So our equation for the line is

$$\ell(t) = \langle 2, 4, 8 \rangle + \langle 1, 4, 12 \rangle t = \langle 2 + t, 4 + 4t, 8 + 12t \rangle .$$

(Note: I'm calling the line $\ell(t)$ since $\mathbf{r}(t)$ was already in use for the original curve. It doesn't matter what you call it, as long as you're clear about what you're doing.)

Problem 2. *Consider the two vectors* $\mathbf{u} = \langle 1, 2, 3 \rangle$ *and* $\mathbf{v} = \langle -1, -1, -1 \rangle$.

a) *What is the angle between* $\mathbf{u}$ *and* $\mathbf{v}$*?*

We know $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}|\,|\mathbf{v}|\cos\theta$, and so the angle is given by

$$\theta = \cos^{-1}\left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|}\right) = \cos^{-1}\left(\frac{-6}{\sqrt{14}\sqrt{3}}\right).$$

b) *If a triangle has* $(0,0,0)$ *as a vertex, with* $\mathbf{u}$ *and* $\mathbf{v}$ *the two edges from this vertex, what is the vector for the third edge?*

It's $\mathbf{u} - \mathbf{v} = \langle 2,3,4\rangle$ (or the other way, depending on which direction we want the edge vector to point).

c) *What is the area of this triangle?*

Remember that magnitude of the cross product gives the area of the parallelogram spanned by the vectors, and we want half of that.

$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ -1 & -1 & -1 \end{vmatrix} = \langle 1,-2,1\rangle\,.$$

So

$$\text{area} = \frac{1}{2}\,|\langle 1,-2,1\rangle| = \frac{1}{2}\sqrt{6}.$$

Problem 3. *Let* $\mathbf{u} = \langle -13, 2, 1\rangle$ *and* $\mathbf{v} = \langle 0, 21, -2\rangle$*. In what general direction does* $\mathbf{u}\times\mathbf{v}$ *point? (Use the right-hand rule.)*

It's more or less straight down: when we use the right hand rule, the first vector points to the left in the xy-plane (it is almost parallel to $-\mathbf{i}$), while the second points straight ahead (roughly $\mathbf{j}$). The right hand rule says the cross product points down.

Problem 4. *Use the two-path test to explain why the limit*

$$\lim_{(x,y)\to(0,0)} \frac{y^2 - 3x}{y^2 - x}$$

does not exist.

I've trained you to always try the path $y = mx$ first. In this case we get

$$\lim_{x\to 0}\frac{(mx)^2 - 3x}{(mx)^2 - x} = \lim_{x\to 0}\frac{m^2x - 3}{x^2x - 1} = 3,$$

which doesn't depend on m. So it seems like the test isn't going to help us. But there are other paths we might want to try too. In this case, we can use the path $x = 0$, and we get

$$\lim_{y\to 0}\frac{y^2}{y^2} = 1,$$

which isn't the same. So the answer actually does depend on the path, even though $y = mx$ all give the same answer. The two-path test tells us the limit doesn't exist.

Problem 5. *A moving particle has* $\mathbf{v}(t) = \langle -\sin t, \cos t, 1\rangle$.

a) *What is the tangent vector to its path at* $t = 0$*?*
That's just $\mathbf{v}(0) = \langle 0, 1, 1\rangle$ (if I'd given you $\mathbf{r}(t)$ instead, you'd first calculate $\mathbf{v}(t) = \mathbf{r}'(t)$, and then plug in 0 to that).

b) *Find the length of the curve from* $t = 0$ *to* $t = 2\pi$*. What is the tangent vector at* $t = 0$*?*
The length is

$$\ell = \int_0^{2\pi} |\mathbf{v}(t)|\, dt = \int_0^{2\pi} \sqrt{(-\sin t)^2 + (\cos t)^2 + (1)^2}\, dt = \int_0^{2\pi} \sqrt{2} = 4\pi.$$

c) *Suppose that* $\mathbf{r}(0) = \langle 0, 0, 0\rangle$*. Find a formula for* $\mathbf{r}(t)$*.*
We know $\mathbf{r}(t) = \int \mathbf{v}(t) + \mathbf{C}$, and so

$$\mathbf{r}(t) = \langle \cos t + c_1, \sin t + c_2, t + c_3\rangle .$$

Plugging in $t = 0$, we get $\langle 0, 0, 0\rangle$ $\langle 1 + c_1, c_2, c_3\rangle$, and so $c_1 = -1$, $c_2 = 0$, and $c_3 = 0$. Thus

$$\mathbf{r}(t) = \langle \cos t - 1, \sin t, t\rangle .$$

Problem 6. *An object sits at* $(1, 1)$ *on a surface sloped* $45°$*. The gravitational force is* $\mathbf{g} = \langle 0, -10\rangle$.

a) *What is the component of the gravitational force parallel to the surface?*
We want the component in the direction $\mathbf{u} = \langle -1, -1\rangle$. This is given by

$$\mathrm{proj}_{\mathbf{u}}\, \mathbf{g} = \left(\frac{\mathbf{g}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u} = \frac{10}{2}\langle -1, -1\rangle = \langle -5, -5\rangle .$$

b) *The object slides down the slope to the point* $(0, 0)$*. Find the work done by gravity.*
It's just $W = \mathbf{F}\cdot\mathbf{d}$, where $\mathbf{d}$ is the displacement. This is $\langle 0, -10\rangle \cdot \langle -1, -1\rangle = 10$ (Joules, perhaps).

Problem 7. *Let* $f(x, y) = x^2y + 3x - 2$.

a) *Find the gradient* ∇f*.*
I got

$$\nabla f = \langle f_x, f_y\rangle = \left\langle 2xy + 3, x^2\right\rangle .$$

b) *Find the direction of fastest increase at the point* $(x, y) = (0, 1)$*. What is the rate of increase?*
Fastest increase occurs in the direction of the gradient, which is $\nabla f(1, 1) = \langle 5, 1\rangle$. The rate of increase (i.e. the directional derivative in this direction) is given by the length of the gradient, which is $\sqrt{26}$.

Problem 8. *What is the domain of the function* $f(x, y) = \ln(x^2 + y^2 - 4)$*? Sketch some level curves.*

The domain is everywhere that the formula is going to make sense. The only thing that can go wrong is that we try to take the logarithm of a negative number, which will happen if $x^2 + y^2 - 4 < 0$. This means that $x^2 + y^2 < 4$, which is the inside of a circle of radius 2.

That's where the function *isn't* defined – the domain is everywhere that it is, which means it's the area outside of a circle of radius 2 centered at the origin.

The level curve at $z = 2$ is all (x, y) with $\ln(x^2 + y^2 - 4) = 2$, so $x^2 + y^2 - 4 = e^2$. This is a circle of radius $\sqrt{4 + e^2}$; all other level curves are also circles too.