Math 210 (Lesieutre)
12.7: Tangent planes and linear approximation
February 17, 2017

Problem 1. *Let's start with a tangent plane to a sphere.*

a) *Write down the equation $f(x, y, z)$ for a sphere with center $(1, 2, 3)$ and radius 3.*
We want the sphere to be

$$(x-1)^2 + (y-2)^2 + (z-3)^2 = 25.$$

b) *One point on the sphere is $(a, b, c) = (2, 0, 5)$. Compute the gradient ∇f, and evaluate $\nabla f(a, b, c)$.*
The gradient is just

$$\nabla f = \langle 2(x-1), 2(y-2), 2(z-3) \rangle .$$

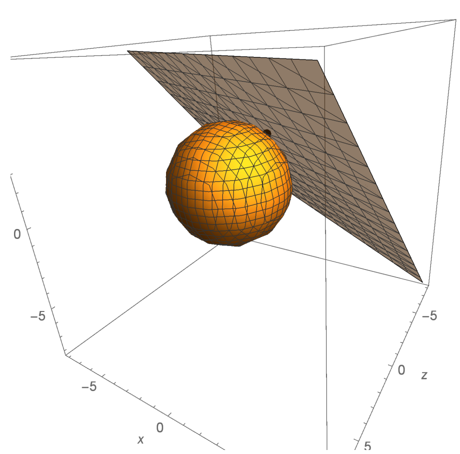
Plugging in the points in question, we get

$$\nabla f(2, 0, 5) = \langle 2, -4, 4 \rangle .$$

c) *Use your answer to write down the equation for the tangent plane to the sphere at (a, b, c).*
The normal direction is $\langle 2, -4, 4 \rangle$, and it goes through the point $(2, 0, 5)$. Using the regular old formula for the equation of a sphere, we get

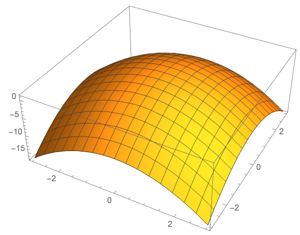
$$2(x-2) - 4(y-0) + 4(z-5) = 0.$$

d) *Try to plot the sphere and the plane and convince yourself that this answer is reasonable.*
I cheated and used a computer. Things look good, however:



(Note: I've rotated the picture somewhat to give us a better view, so yours might look different.)

Problem 2. *Let* $f(x, y) = 1 - x^2 - y^2$.



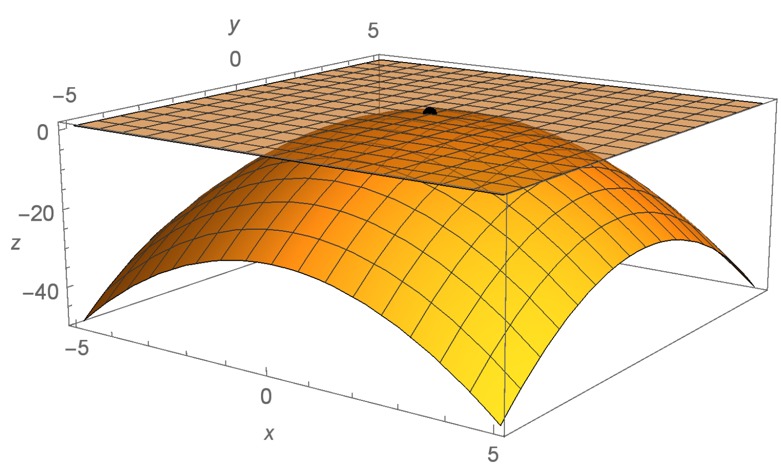
a) *Find the tangent plane to the graph at* $(x, y) = (0, 0)$. *Does your answer make sense?*
The formula is

$$z = f_x(a, b)(x - a) + f_y(a, b)(y - b) + f(a, b)$$

In our situation,

$$\begin{aligned} f(x, y) &= 1 - x^2 - y^2 \\ f(0, 0) &= 0 \\ f_x(x, y) &= -2x \\ f_y(x, y) &= -2y \\ f_x(0, 0) &= 0 \\ f_y(0, 0) &= 0 \end{aligned}$$

So the plane is $z = 0(x - 0) + 0(y - 0) + 1$ i.e. a horizontal plane $z = 1$. This makes sense, based on the graph: it's the tangent plane at the very top of the graph.



b) *Find the tangent plane to the graph at* $(x, y) = (1, -1)$. *Does your answer make sense?*

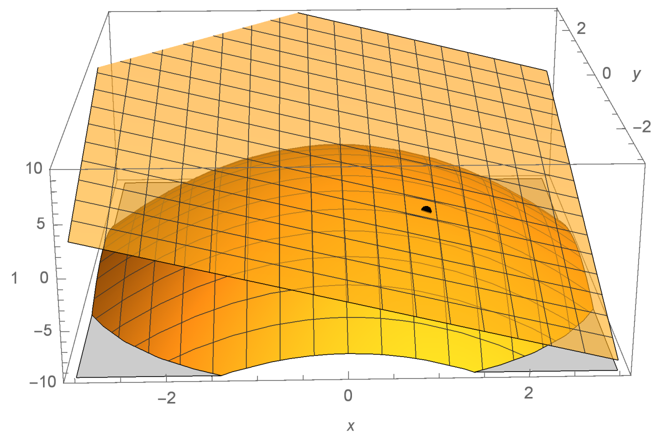
This time we get

$$f(1,-1) = -1$$
$$f_x(1,-1) = -2$$
$$f_y(1,-1) = 2$$

The plane is

$$z = -1 - 2(x-1) + 2(y+1)$$

Seems plausible. Here's a graph:



Problem 3. *Consider the function $f(x,y) = \frac{1}{x^2+y^2}$. Use a linear approximation to approximate the value of $f(1.1, 1.9)$.*

$(1.1, 1.9)$ is close to the much cleaner value $(1,2)$, so we're going to use $(a,b) = (1,2)$ in the approximation formula. First, let's figure out what the formula actually gives is in this case. The derivative needs a little thought. To get f_x, we treat y as a constant C, and then our function is $g(h(x))$ where $g(x) = 1/x$ and $h(x) = x^2 + C$. Notice that $g'(x) = -1/x^2$ and $h'(x) = 2x$, and so according to the chain rule the derivative is $g'(h(x))h'(x) = -1/(x^2 + C)^2 \cdot 2x = -\frac{2x}{(x^2+y^2)^2}$.

$$f(x,y) = \frac{1}{x^2+y^2}$$
$$f_x = -\frac{2x}{(x^2+y^2)^2}$$
$$f_y = -\frac{2y}{(x^2+y^2)^2}$$

Now plug in the values $(1, 2)$:

$$f(1,2) = \frac{1}{5}$$
$$f_x(1,2) = -\frac{2}{25}$$
$$f_y(1,2) = -\frac{4}{25}.$$

The linear approximation is then

$$L(x, y) \approx \frac{1}{5} - \frac{2}{25}(x-1) - \frac{4}{25}(y-2).$$

We want $x = 1.1$ and $y = 1.9$, in which case this gives

$$L(x, y) \approx \frac{1}{5} - \frac{2}{25}(0.1) - \frac{4}{25}(-0.1) = \frac{50}{250} - \frac{2}{250} + \frac{4}{250} = \frac{52}{250} = 0.208.$$

How did we do? The true value is about 0.207469, so we're off by 0.00053.

Problem 4. *A cylinder has radius* 2 *and height* 3*. Suppose that the radius and height each increase by* 0.1*. Approximately how much will the volume change?*

The volume is given by $V(r, h) = \pi r^2 h$. We have $dV = V_r(a, b)\, dr + V_h(a, b)\, dh$. In this case $V_r = 2\pi rh$ so $V_r(2, 3) = 12\pi$ and $V_h = \pi r^2$ so $V_h(2, 3) = 4\pi$. Then

$$dV = V_r(, b)\, dr + V_h(a, b)\, dh = (12\pi)(0.1) + (4\pi)(0.1) = 1.6\pi.$$

Let's check it. In fact the cylinder has volume 12π, and the new cylinder has volume 13.671π. The increase is 1.671π, which is just about what we expected. (This "true increase" is what the book calls Δz.)