Math 210 (Lesieutre)
12.8: Max/min, continued
February 22, 2017

Problem 1. *For each of the following functions, find the critical points. Pick one, and determine whether it is a maximum, minimum, or saddle point.*

a) $x^4 + y^4 - 16xy$
We have $f_x = 4x^3 - 16y$ and $f_y = 4y^3 - 16x$. The first of these then gives $y = x^3/4$, and plugging this into the second, we obtain

$$\begin{aligned} 4\left(\frac{x^3}{4}\right)^3 - 16x &= 0 \\ 4\frac{x^9}{64} - 16x &= 0 \\ \frac{x^9}{16} - 16x &= 0 \\ x^9 - 256x &= 0 \\ x(x^8 - 256) &= 0 \end{aligned}$$

The second factor $x^8 - 256$ is 0 if $x = 2$ or $x = -2$ (there are six other solutions over the complex numbers, but we're only worried about the real solutions). If $x = 0$, then $y = x^3/4 = 0$; if $x = 2$, then $y = 2$; if $x = -2$, then $y = -2$. So there are three critical points: $(2, 2)$, $(0, 0)$, and $(-2, -2)$.

We need to figure out which of these are maxes and which are mins. To do that we're going to need the second derivatives, so we might as well get it over with and calculate them now.

$$\begin{aligned} f_{xx} &= 12x^2, \\ f_{xy} &= -16, \\ f_{yy} &= 12y^2. \end{aligned}$$

At $(2, 2)$, this gives $f_{xx}(2, 2) = 48$, $f_{yy}(2, 2) = 48$, and $f_{xy}(2, 2) = -16$. Thus $D(2, 2) = (48)(48) - (-16)^2 = 2048$. The test tells us that this is a minimum.

At $(-2, -2)$, this gives $f_{xx}(-2, -2) = 48$, $f_{yy}(-2, -2) = 48$, and $f_{xy}(-2, -2) = -16$. Thus $D(-2, -2) = (48)(48) - (-16)^2 = 2048$. The test tells us that this is a minimum.

At $(0, 0)$, this gives $f_{xx}(0, 0) = 0$, $f_{yy}(0, 0) = 0$, and $f_{xy}(0, 0) = -16$. Thus $D(0, 0) = (0)(0) - (-16)^2 = -256$. So that's a saddle point.

b) $f(x, y) = x^2 - 2x + y^2 - 4y + xy + 5$
We have

$$\begin{aligned} f_x &= 2x - 2 + y, \\ f_y &= 2y - 4 + x. \end{aligned}$$

We need both of these to be 0, so $2x + y = 2$ and $x + 2y = 4$. Solving the linear system using whatever your favorite method is (probably just eliminating one of the variables), we get $x = 0$ and $y = 2$. This is the only critical point.

Is it a maximum or a minimum? Well,

$$\begin{aligned} f_{xx} &= 2 \\ f_{xy} &= f_{yx} = 1 \\ f_{yy} &= 2 \end{aligned}$$

look at the determinant:

$$\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3.$$

It's either or maximum or a minimum. To know which, we have $f_{xx} = 2$, so it's a minimum.

c) $g(x, y) = x^2 y e^{-x^2 - y^2}$.

This is going to be fun (not really)! Let's get all of the derivatives out of the way right off the bat.

$$\begin{aligned} g_x &= x^2 y e^{-x^2-y^2}(-2x) + 2xy e^{-x^2-y^2} = 2(2xy - 2x^3 y)e^{-x^2-y^2}, \\ g_y &= x^2 e^{-x^2-y^2} + (x^2 y)e^{-x^2-y^2}(-2y) = (x^2 - 2x^2y^2)e^{-x^2-y^2}. \end{aligned}$$

So critical points happen whenever $g_x = g_y = 0$. Since $e^{-x^2-y^2}$ is never 0, we need to solve $2(2xy - 2x^3y) = 0$ and $x^2 - 2x^2y^2 = 0$. So $xy(1 - x^2) = 0$ and $x^2(1 - 2y^2) = 0$.

There are two possibilities: either $x = 0$ and y is another, or $y = \pm\frac{1}{\sqrt{2}}$ and $x = \pm 1$.

I'm not actually going to figure out what are the max and mins and saddle points. The point of this problem is mostly to warn you: sometimes there are infinitely many critical points!

Problem 2. *Find the point on the surface $x^2 - yz = 1$ which is closest to the origin. (Hint: minimize the square of the distance, instead of the distance itself).*

The square of the distance is $x^2 + y^2 + z^2$. Since $x^2 = 1 + yz$ on the surface, we want to minimize $1 + yz + y^2 + z^2$ (a function of the independent variables y and z). So we'll use the second-derivative test:

$$\frac{\partial f}{\partial y} = 2y + z, \qquad \frac{\partial f}{\partial z} = y + 2z.$$

These vanish simultaneously at $(0, 0)$, where $x = \pm 1$. So the two critical points of the distance are $(1, 0, 0)$ and $(-1, 0, 0)$. The second derivatives are

$$D(y, z) = \begin{vmatrix} f_{yy} & f_{yz} \\ f_{zy} & f_{zz} \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3$$

As this is positive, and $f_{yy} > 0$, this is a minimum as expected.

Problem 3. *Find the maximum value of the function* $f(x,y) = (x-1)^2 + y^2$ *on or inside a circle of radius* 2 *centered at* $(0,0)$.

First we find the critical points on the interior, i.e. on the inside of the circle. $f_x = 2(x-1)$ and $f_y = 2y$, so the unique critical point is $(1,0)$. By inspection it is a minimum, and so the maximum of the circle is going to be on the boundary.

To find the maximum value on the boundary, use a parametrization $\mathbf{r}(t) = (2\cos t, 2\sin t)$. This parametrizes the circle, and so we just need to find the t for which $f(\mathbf{r}(t))$ is as large as possible. Plugging in, we want to maximize

$$(2\cos t - 1)^2 + (2\sin t)^2 = 4\cos^2 t - 4\cos t + 1 + 4\sin^2 t = 5 - 4\cos t.$$

This has a maximum at $t = \pi$, when $\cos t = -1$. The corresponding point is $\gamma(\pi) = (-2,0)$ and the value of f there is 3.

Problem 4. *Find the maximum and minimum values of the function* $f(x,y) = 5 - (x-1)^2 - (y-1)^2$ *on a triangle with vertices* $(0,0)$*,* $(8,0)$*, and* $(0,4)$.

First we need to find the critical points. That's the easy part. We have

$$f_x = -2(x-1),$$
$$f_y = -2(y-1).$$

The critical points are where both of these vanish. There is only one: the point $(x,y) = (1,1)$. This is inside the triangle in question, as needed.

Is that point a maximum or a minimum? Maybe you can guess from the formula (it sure looks like a max...), but the way to check is to use the second derivative test with the Hessian.

The second derivatives are

$$f_{xx} = -2,$$
$$f_{xy} = 0,$$
$$f_{yy} = -2.$$

At $(1,1)$, we have $f_{xx}(1,1) = 2$, $f_{xy}(1,1) = 0$, $f_{yy}(1,1) = 2$. Thus $D(1,1) = (-2)(-2) - (0)(0) = 4$, which is positive. Since $f_{xx}(1,1) > 0$, the point is a maximum. For future comparison, notice that $f(1,1) = 5$.

However, we still don't actually know that it's a global maximum: maybe the function achieves an even bigger value somewhere, but it's on the edge, and so it isn't a critical point. So we need to find the maxima and minima going around the outside of the triangle.

We have to check each edge. To do that, we'll parametrize it, and then solve for the value(s) of t that maximize and minimize the function in question. Let's start with the edge from $(0,4)$ to $(8,0)$. It's parametrized by $\mathbf{r}_1(t) = \langle 8t, 4-4t\rangle$, where $0 \leq t \leq 1$ (here I used our usual strategy for parametrizing a line. Then we get:

$$f(\mathbf{r}_1(t)) = 5 - (x-1)^2 - (y-1)^2 = 5 - (3-4t)^2 - (-1+8t)^2 = -80t^2 + 40t - 5.$$

We now do max/min, math 180-style. The derivative is $-160t+40$, which is 0 when $t = 1/4$. This corresponds to the point $(2,3)$, for which $f(\mathbf{r}_1(1/4)) = 0$. Since $f''(\mathbf{r}_1(t)) < 0$, it's a local max, and we find $f(\mathbf{r}_1(5/2)) = \frac{1}{2}$. We also need to check the two endpoints: $t = 0$ and $t = 4$. At $t = 0$, we're at the point $\mathbf{r}_1(0) = (0,4)$, and $f(\mathbf{r}_1(0)) = -5$. At $t = 4$, it's $(4,0)$, and $f(\mathbf{r}_1(4)) = -5$ too.

That's only one edge. There are two more. Let's go from $(0,0)$ to $(8,0)$ now. This one has $\mathbf{r}_2(t) = \langle 8t, 0\rangle$ with $0 \le t \le 1$. Then $f(\mathbf{r}_2(t)) = 5 - (8t-1)^2 - (-1)^2 = 4 - (8t-1)^2 = -64t^2 + 16t + 3$. What are the extrema? Well, $\frac{d}{dt} f(\mathbf{r}_2(t)) = 16 - 128t$, which vanishes for $t = 1/8$, which is the point $(1,0)$. Here the function has the value 4. There are also the endpoints $\mathbf{r}_2(0) = (0,0)$ and $\mathbf{r}_2(1) = (8,0)$, where the values are respectively 3 and -45.

The last edge goes from $(0,0)$ to $(0,4)$, and is parametrized by $\mathbf{r}_3(t) = \langle 0, 4t\rangle$. Then $f(\mathbf{r}_3(t)) = 5 - (4t-1)^2 - (-1)^2 = -16t^2 + 8t + 3$. Then $\frac{d}{dt} f(\mathbf{r}_3(t)) = 8 - 32t$, which vanishes for $t = 1/4$, corresponding to the point $\mathbf{r}_3(1/4) = (0,1)$, for which $f(\mathbf{r}_3(1/4)) = 4$. The two endpoints are $(0,0)$ and $(0,4)$, which are already on the list.

All told, the possible extrema come from the edges and the critical points of the 2D function. They're all listed below.

Part of R	Possible extremum	Value	Type
Interior	$(1,1)$	5	Max
Edge #1	$(2,3)$	0	Min
	$(8,0)$	-45	?
	$(0,4)$	-5	?
Edge #2	$(1,0)$	4	Min
	$(0,0)$	3	?
	$(8,0)$	-45	?
Edge #3	$(0,1)$	4	Min
	$(0,0)$	3	?
	$(0,4)$	-5	?

(The ?'s indicate that at the corners of the region, we don't really know if the points are going to be maxima or minima, because we can't use the second derivative test. We just have to compare the values against the other values.

Those are the candidates. To see the actual global max and min on the region, look for the biggest and smallest numbers. The global max occurs at the critical point $(1,1)$, which the value is 5. The global min is at $(8,0)$, where the value is -45.