Math 210 (Lesieutre)
12.8: Max/min, continued
February 25, 2017

Problem 1. *An airline will let you carry on any rectangular bag for which the sum of the dimensions x, y, and z is less than 60. Suppose that you want to bring a bag with the largest possible volume. To find the appropriate x, y, and z, what function should you maximize, and on what region?*

The volume is xyz. We can assume that $z = 60 - x - y$, since otherwise we could make z bigger without breaking the rules. Then the function we should maximize is $xy(60 - x - y)$. The region is where $x \geq 0$, $y \geq 0$, and $x + y \leq 60$, which is a triangle.

Problem 2. *Find the maximum and minimum values of the function $f(x, y) = 5 - (x - 1)^2 - (y - 1)^2$ on a triangle with vertices $(0, 0)$, $(8, 0)$, and $(0, 4)$.*

First we need to find the critical points. That's the easy part. We have

$$f_x = -2(x - 1),$$
$$f_y = -2(y - 1).$$

The critical points are where both of these vanish. There is only one: the point $(x, y) = (1, 1)$. This is inside the triangle in question, as needed.

Is that point a maximum or a minimum? Maybe you can guess from the formula (it sure looks like a max…), but the way to check is to use the second derivative test with the Hessian.

The second derivatives are

$$f_{xx} = -2,$$
$$f_{xy} = 0,$$
$$f_{yy} = -2.$$

At $(1, 1)$, we have $f_{xx}(1, 1) = 2$, $f_{xy}(1, 1) = 0$, $f_{yy}(1, 1) = 2$. Thus $D(1, 1) = (-2)(-2) - (0)(0) = 4$, which is positive. Since $f_{xx}(1, 1) > 0$, the point is a maximum. For future comparison, notice that $f(1, 1) = 5$.

However, we still don't actually know that it's a global maximum: maybe the function achieves an even bigger value somewhere, but it's on the edge, and so it isn't a critical point. So we need to find the maxima and minima going around the outside of the triangle.

We have to check each edge. To do that, we'll parametrize it, and then solve for the value(s) of t that maximize and minimize the function in question. Let's start with the edge from $(0, 4)$ to $(8, 0)$. It's parametrized by $\mathbf{r}_1(t) = \langle 8t, 4 - 4t \rangle$, where $0 \leq t \leq 1$ (here I used our usual strategy for parametrizing a line. Then we get:

$$f(\mathbf{r}_1(t)) = 5 - (x-1)^2 - (y-1)^2 = 5 - (3-4t)^2 - (-1+8t)^2 = -80t^2 + 40t - 5.$$

We now do max/min, math 180-style. The derivative is $-160t+40$, which is 0 when $t = 1/4$. This corresponds to the point $(2,3)$, for which $f(\mathbf{r}_1(1/4)) = 0$. Since $f''(\mathbf{r}_1(t)) < 0$, it's a local max, and we find $f(\mathbf{r}_1(5/2)) = \frac{1}{2}$. We also need to check the two endpoints: $t = 0$ and $t = 4$. At $t = 0$, we're at the point $\mathbf{r}_1(0) = (0,4)$, and $f(\mathbf{r}_1(0)) = -5$. At $t = 4$, it's $(4,0)$, and $f(\mathbf{r}_1(4)) = -5$ too.

That's only one edge. There are two more. Let's go from $(0,0)$ to $(8,0)$ now. This one has $\mathbf{r}_2(t) = \langle 8t, 0\rangle$ with $0 \le t \le 1$. Then $f(\mathbf{r}_2(t)) = 5 - (8t-1)^2 - (-1)^2 = 4 - (8t-1)^2 = -64t^2 + 16t + 3$. What are the extrema? Well, $\frac{d}{dt} f(\mathbf{r}_2(t)) = 16 - 128t$, which vanishes for $t = 1/8$, which is the point $(1,0)$. Here the function has the value 4. There are also the endpoints $\mathbf{r}_2(0) = (0,0)$ and $\mathbf{r}_2(1) = (8,0)$, where the values are respectively 3 and -45.

The last edge goes from $(0,0)$ to $(0,4)$, and is parametrized by $\mathbf{r}_3(t) = \langle 0, 4t\rangle$. Then $f(\mathbf{r}_3(t)) = 5 - (4t-1)^2 - (-1)^2 = -16t^2 + 8t + 3$. Then $\frac{d}{dt} f(\mathbf{r}_3(t)) = 8 - 32t$, which vanishes for $t = 1/4$, corresponding to the point $\mathbf{r}_3(1/4) = (0,1)$, for which $f(\mathbf{r}_3(1/4)) = 4$. The two endpoints are $(0,0)$ and $(0,4)$, which are already on the list.

All told, the possible extrema come from the edges and the critical points of the 2D function. They're all listed below.

Part of R	Possible extremum	Value	Type
Interior	$(1,1)$	5	Max
Edge #1	$(2,3)$	0	Min
	$(8,0)$	-45	?
	$(0,4)$	-5	?
Edge #2	$(1,0)$	4	Min
	$(0,0)$	3	?
	$(8,0)$	-45	?
Edge #3	$(0,1)$	4	Min
	$(0,0)$	3	?
	$(0,4)$	-5	?

(The ?'s indicate that at the corners of the region, we don't really know if the points are going to be maxima or minima, because we can't use the second derivative test. We just have to compare the values against the other values.

Those are the candidates. To see the actual global max and min on the region, look for the biggest and smallest numbers. The global max occurs at the critical point $(1,1)$, which the value is 5. The global min is at $(8,0)$, where the value is -45.

Problem 3. *Consider the function* $f(x,y) = e^{-x^2-2y^2}$.

a) *Find the maximum and minimum on or inside a square with vertices* $(\pm 2, \pm 2)$.

Step 1: Find the critical points.

We have

$$f_x(x,y) = -e^{-x^2-2y^2}(-2x) = 2xe^{-x^2-2y^2},$$
$$f_y(x,y) = -e^{-x^2-2y^2}(-4y) = 4ye^{-x^2-2y^2}.$$

How can the first of these be 0? It's only if $2x = 0$, since $e^{-x^2-2y^2}$ can't possibly be 0. How about the second? Only if $4y = 0$, for the same reason. So the only critical point is $(0,0)$.

We'll need to know the second partials to determine the types of the critical point. This is a little more messy, since we have to use the product rule.

$$f_{xx}(x,y) = -2e^{-x^2-2y^2} + 4x^2e^{-x^2-2y^2}x^2$$
$$f_{xy}(x,y) = 8xye^{-x^2-2y^2}$$
$$f_{yy}(x,y) = -4e^{-x^2-2y^2} + 16y^2e^{-x^2-2y^2}$$

At $(0,0)$, the values of these things are respectively -2, 0, and -4. Then $D(0,0) = (-2)(-4) - 0^2 = 8$, which is positive, and since $f_{xx}(0,0) < 0$, we conclude that the point is a maximum.

The annoying part is that the maximum or minimum could be on an edge, and there are four of them. Here are the parametrizations:

$$\mathbf{r}_1(t) = \langle 2, -2+4t \rangle \quad \text{(top)}$$
$$\mathbf{r}_2(t) = \langle -2, -2+4t \rangle \quad \text{(bottom)}$$
$$\mathbf{r}_3(t) = \langle -2+4t, -2 \rangle \quad \text{(left)}$$
$$\mathbf{r}_4(t) = \langle -2+4t, 2 \rangle \quad \text{(right)}$$

Let's do $\mathbf{r}_1(t)$ in detail. We need to plug this in to $f(t)$ and think of it as a function of the single variable t, which we then find max/min for using the usual single-variable calculus strategies. Plugging in, we get

$$f(\mathbf{r}_1(t)) = e^{-4-2(-2+4t)^2} = e^{-12+32t-32t^2}.$$

The derivative is $\frac{d}{dt}f(\mathbf{r}_1(t)) = (-64t+32)e^{-12+32t-32t^2}$, which is 0 when $t = 1/2$. This corresponds to the point $(2,0)$, and we find $f(2,0) = 1/e^4$. This is a local maximum (on the edge). We also need to check the endpoints, which are $(2,2)$ and $(2,-2)$. Both of these give $1/e^{12}$.

Now we need to do the second edge. We find that $f(\mathbf{r}_2(t)) = e^{-12+32t-32t^2}$, which is the same thing that we got with $\mathbf{r}_1(t)$. There is again a maximum at $t = 1/2$, which is the point $(-2,0)$, and the value is again $1/e^4$. The two endpoints $(-2,-2)$ and $(-2,2)$ give values of $1/e^{12}$.

Now the third edge $\mathbf{r}_3(t) = \langle -2 + 4t, -2 \rangle$ gives us $f(\mathbf{r}_3(t)) = e^{-12+16t-16t^2}$. The derivative is $\frac{d}{dt} f(\mathbf{r}_3(t)) = e^{-12+16t-16t^2}(16 - 32t)$, which *again* has a max at $t = 1/2$, which this time is the point $(0, -2)$, where we get $1/e^8$.

It's getting very late, so I'm not going to write down the fourth edge. Sorry! Take my word for it that it works the same as the third one, or email me if you want me to add some more details.

Part of R	Possible extremum	Value	Type
Interior	$(0,0)$	1	Max
Edge #1	$(2,0)$	$1/e^4$	Max
	$(2,-2)$	$1/e^{12}$	?
	$(2,2)$	$1/e^{12}$	?
Edge #2	$(-2,0)$	$1/e^4$	Max
	$(-2,-2)$	$1/e^{12}$	?
	$(-2,2)$	$1/e^{12}$	?
Edge #3	$(0,2)$	$1/e^8$	Max
	$(-2,2)$	$1/e^{12}$	?
	$(2,2)$	$1/e^{12}$	?
Edge #4	$(0,-2)$	$1/e^8$	Max
	$(-2,-2)$	$1/e^{12}$	?
	$(2,-2)$	$1/e^{12}$	?

The global max is $(0, 0)$, where the function is $f(0, 0) = 1$. The global min is at each of the four corners $(\pm 2, \pm 2)$, where f takes the value $1/e^{12}$.

b) *Find the maximum of the same function on the unit circle.*

The strategy is the same, and this one is actually a little easier, since a circle only has one edge. Parametrize it by $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$. Plugging that in, we get

$$f(\mathbf{r}(t)) = e^{-\cos^2 t - 2\sin^2 t} = e^{-\frac{3}{2} + \frac{1}{2}\cos(2t)}$$

(I used a trig identity here, but you'd get the same answer even without it.) We have

$$\frac{d}{dt} f(\mathbf{r}(t)) = e^{-\frac{3}{2} + \frac{1}{2}\cos(2t)}(-2\sin(2t)).$$

The critical points are $t = 0, \pi/2, \pi$, and $3\pi/2$. The values of the function are

t	$f(\mathbf{r}(t))$
0	e^{-1}
$\pi/2$	e^{-2}
π	e^{-1}
$3\pi/2$	e^{-2}

The critical points are all either on the edge, or in the interior of the unit disk, where we already found that the only critical point is $(0, 0)$. Hence the max is at $(0, 0)$, where $f(0, 0) = 1$, and the minimum occurs on the boundary at $\mathbf{r}(\pi/2) = (0, 1)$ and $\mathbf{r}(3\pi/2) = (0, -1)$. At these points, the value of the function is e^{-2}.