Math 210 (Lesieutre)
12.9: Lagrange multipliers
February 28, 2017

Problem 1. *You want to find a rectangle with perimeter* 16 *and area as large as possible.*

a) *Convert this into a constrained optimization problem: let x and y be the two sides of the rectangle. What function $f(x, y)$ are you trying to maximize? What constraint must be satisfied by x and y?*
We are trying to maximize $f(x, y) = xy$ subject to $g(x, y) = 2x + 2y = 16$.

b) *Use the method of Lagrange multipliers to find the maximum value of the function.*
We have

$$\nabla f = \langle y, x \rangle$$
$$\nabla g = \langle 2, 2 \rangle$$

The equations are $\nabla f(x, y) = \lambda \nabla g(x, y)$ and $g(x, y) = 0$, which gives three equations in three variables:

$$y = 2\lambda$$
$$x = 2\lambda$$
$$2x + 2y = 16$$

The second equation becomes $2(2\lambda)+2(2\lambda) = 16$, so that $\lambda = 2$. Then $x = 4$ and $y = 4$ must be the maximum. The value of the function is given by $f(4, 4) = 16$. Hence the maximum possible area is achieved when the function is a square.

Problem 2. *Find the maximum value of the function $f(x, y) = y^2 - 4x^2$ subject to the constraint $g(x, y) = x^2 + 2y^2 = 4$.*

In this case, we have

$$\nabla f(x, y) = \langle -8x, 2y \rangle$$
$$\nabla g(x, y) = \langle 2x, 4y \rangle$$

Our three equations come from $\nabla f(x, y) = \lambda \nabla g(x, y)$, so that

$$-8x = \lambda 2x$$
$$2y = \lambda 4y$$
$$x^2 + 2y^2 = 4$$

This is a fairly typical set-up for one of these problems. The equations aren't so hard to solve, but you need to be extremely careful not to forget that the variables can be 0.

If x is not 0, then dividing the first equation through by x gives $\lambda = -4$. Then the second equation gives $2y = -16y$, which means that $y = 0$. The third then says that $x^2 = 4$, so $x = -2$ or $x = 2$. This gives two points: $(x, y) = (-2, 0)$ and $(x, y) = (2, 0)$. We have $f(-2, 0) = -16$ and $f(2, 0) = -16$.

If x is 0, then the first equation is true no matter what λ is. The third equation reads $2y^2 = 4$, so $y = \pm\sqrt{2}$. Since y is nonzero, the second equation just turns into $\lambda = 1/2$ (but again, we don't really care what λ is). So we have two more solutions: $(x, y) = (0, -\sqrt{2})$ and $(x, y) = (0, \sqrt{2})$. Plugging in we get $f(0, -\sqrt{2}) = 2$ and $f(0, \sqrt{2}) = 2$. So the function is maximized at $(0, -\sqrt{2})$ and $(0, \sqrt{2})$, where the value is 2, and minimized at $(2, 0)$ and $(-2, 0)$, where the value is -16.

Problem 3. *Use Lagrange multipliers to find the point on the parabola $y = x^2 - 1$ which is closest to the origin.*

We want to minimize the distance function $h(x, y) = \sqrt{x^2 + y^2}$. Nobody likes square roots, so notice that we can just as well minimize the distance squared, which is $f(x, y) = x^2 + y^2$. This is subject to the constraint $y = x^2 - 1$. This is better written as $g(x, y) = y - x^2 + 1 = 0$.

We have

$$\nabla f(x, y) = \langle 2x, 2y \rangle$$
$$\nabla g(x, y) = \langle -2x, 1 \rangle$$

Our equations are thus $2x = -2x\lambda$, $2y = \lambda$, and $y - x^2 + 1 = 0$. Assume first that $x \neq 0$. Then $\lambda = -1$ from the first equation, so $y = -1/2$. Then $(-1/2) - x^2 + 1 = 0$ so that $x = \pm\sqrt{1/2}$. So we get two points: $\left(\frac{\sqrt{2}}{2}, -\frac{1}{2}\right)$. The square of the distance at either one is $f\left(\frac{\sqrt{2}}{2}, -\frac{1}{2}\right) = \frac{3}{4}$.

If $x = 0$, then the third equation gives $y = -1$. The distance here is 1. So the closest points were the first two that we found.

Problem 4. *You are making a open-top drawer out of wood. The material for the sides and back costs \$2 per square foot, and material for the bottom costs \$1, and the material for the front costs \$4.*

a) *Suppose that the dimensions of the drawer are x (side to side), y (top to bottom), and z (front to back). What is the total cost of the materials?*

It should be

$$\begin{aligned} \text{cost} &= \text{cost}_{\text{front}} + \text{cost}_{\text{back}} + \text{cost}_{\text{sides}} + \text{cost}_{\text{bottom}} \\ &= 4xy + 2xy + 2(2yz) + xz = 6xy + 4yz + xz. \end{aligned}$$

b) *What are the dimensions of the cheapest drawer with volume* 24 *cubic feet?*

We want to minimize $f(x, y, z) = 6xy + 4yz + xz$ subject to the constraint $xyz = 24$. The gradients are

$$\nabla f(x, y, z) = \langle 6y + z, 6x + 4z, 4y + x \rangle$$
$$\nabla g(x, y, z) = \langle yz, xz, xy \rangle$$

Our four equations are

$$(6y + z) = \lambda(yz)$$
$$(6x + 4z) = \lambda(xz)$$
$$(4y + x) = \lambda(xy)$$
$$xyz = 24$$

After some painful algebra, you'll get that $x = 4$, $y = 1$, $z = 6$, and $\lambda = 2$. So the minimum is a drawer that's $4 \times 1 \times 6$.