Math 210 (Lesieutre)
12.9, Lagrange multipliers, and a little 13.1
March 1, 2017

Problem 1. *Find the maximum value of the function $f(x,y) = x+y$ subject to the constraint $x^2+y^2=1$.*

First let's come up with the equations. The functions are $f(x,y) = x+y$ and $g(x,y) = x^2+y^2-1$.

$$\nabla f(x,y) = \langle 1,1 \rangle$$
$$\nabla g(x,y) = \langle 2x, 2y \rangle .$$

The equations are then $\langle 1,1 \rangle = \lambda \langle 2x, 2y \rangle$, so

$$2x\lambda = 1,$$
$$2y\lambda = 1,$$
$$x^2+y^2 = 1.$$

Notice that λ can't possibly be 0: then the first equation wouldn't have any solution. Assuming $\lambda \neq 0$, the first two equations give $x = y$, from which the third gives $2x^2 = 1$, i.e. $x = \pm\frac{\sqrt{2}}{2}$. Thus there are two possibilities: $(x,y) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ and $(x,y) = -\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$.

(x,y)	$f(x,y)$
$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$	$\sqrt{2}$
$\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$	$-\sqrt{2}$

The first point is the max, and the second point is the min.

Problem 2. *You are making a open-top drawer out of wood. The material for the sides and back costs* \$2 *per square foot, and material for the bottom costs* \$1*, and the material for the front costs* \$4*.*

a) *Suppose that the dimensions of the drawer are x (side to side), y (top to bottom), and z (front to back). What is the total cost of the materials?*
It should be

$$\begin{aligned} \text{cost} &= \text{cost}_{\text{front}} + \text{cost}_{\text{back}} + \text{cost}_{\text{sides}} + \text{cost}_{\text{bottom}} \\ &= 4xy + 2xy + 2(2yz) + xz = 6xy + 4yz + xz. \end{aligned}$$

b) *What are the dimensions of the cheapest drawer with volume 24 cubic feet?*

We want to minimize $f(x, y, z) = 6xy + 4yz + xz$ subject to the constraint $xyz = 24$. The gradients are

$$\begin{aligned}\nabla f(x, y, z) &= \langle 6y + z, 6x + 4z, 4y + x\rangle \\ \nabla g(x, y, z) &= \langle yz, xz, xy\rangle\end{aligned}$$

Our four equations are

$$\begin{aligned}(6y + z) &= \lambda(yz) \\ (6x + 4z) &= \lambda(xz) \\ (4y + x) &= \lambda(xy) \\ xyz &= 24\end{aligned}$$

After some painful algebra, you'll get that $x = 4$, $y = 1$, $z = 6$, and $\lambda = 2$. So the minimum is a drawer that's $4 \times 1 \times 6$.

Problem 3. *Compute the following double integrals.*

a)

$$\int_0^1 \int_1^2 xy\, dy\, dx$$

The inner integral is everything from the second integral sign to the $d$whatever.

$$\int_1^2 xy\, dy = \frac{y^2}{2}x\Big|_{y=1}^{2} = \frac{2^2}{2}x - \frac{1^2}{2}x = \frac{3}{2}x.$$

The outer integral is then

$$\int_0^1 \frac{3}{2}x\, dx = \frac{3}{2}\frac{x^2}{2}\Big|_{x=0}^{1} = \frac{3}{4} - 0 = \frac{3}{4}.$$

That's our final answer.

b)

$$\int_0^1 \int_0^2 ye^{xy}\, dx\, dy$$

The inner integral is given by

$$\int_0^2 e^{xy}\, dx = e^{xy}\Big|_{x=0}^{2} = e^{2y} - e^{0y} = e^{2y} - 1.$$

The final answer is

$$\int_0^1 e^{2y} - 1\, dy = \left(\frac{1}{2}e^{2y} - y\right)\Big|_{y=0}^{1} = \left(\frac{1}{2}e^2 - 1\right) - \left(\frac{1}{2} - 0\right) = \frac{1}{2}e^2 - \frac{3}{2}.$$

Notice that if we tried to do it in the other order, it would be tough going: $\int ye^{xy}\, dy$ would require an integration by parts. In some cases, it's even worse: the integral simply can't be done unless the order is right.