Math 210 (Lesieutre)
13.1: Double integrals over rectangular regions
March 3, 2017

Problem 1. *Compute the following double integrals. Sketch the region R over which the integral is being taken.*

a) $\int_0^1 \int_1^2 xy\,dy\,dx$
The inner integral is everything from the second integral sign to the $d$whatever.

$$\int_1^2 xy\,dy = \frac{y^2}{2}x\Big|_{y=1}^{2} = \frac{2^2}{2}x - \frac{1^2}{2}x = \frac{3}{2}x.$$

The outer integral is then

$$\int_0^1 \frac{3}{2}x\,dx = \frac{3}{2}\frac{x^2}{2}\Big|_{x=0}^{1} = \frac{3}{4} - 0 = \frac{3}{4}.$$

That's our final answer.

b) $\int_1^2 \int_0^1 xy\,dx\,dy$
Inner:

$$\int_0^1 xy\,dx = y\frac{x^2}{2}\Big|_{x=0}^{1} = \frac{y}{2}.$$

Outer:

$$\int_1^2 \frac{y}{2}\,dy = \frac{y^2}{4}\Big|_1^2 = \frac{3}{4}.$$

Same answer as before. Coincidence? No – Fubini's theorem. Changing the order of the variables doesn't change the value of the integral.

Problem 2. a) $\int_0^2 \int_0^1 ye^{xy}\,dy\,dx$
The inner integral is $\int_0^1 ye^{xy}\,dy$, which is moderately unpleasant, since we have to integrate by parts. If we change the order of integration, it's a little better:

$$\int_0^1 \int_0^2 ye^{xy}\,dx\,dy$$

The inner integral is given by

$$\int_0^2 ye^{xy}\,dx = e^{xy}\Big|_{x=0}^{2} = e^{2y} - e^{0y} = e^{2y} - 1.$$

The final answer is

$$\int_0^1 e^{2y} - 1\,dy = \left(\frac{1}{2}e^{2y} - y\right)\Big|_{y=0}^{1} = \left(\frac{1}{2}e^2 - 1\right) - \left(\frac{1}{2} - 0\right) = \frac{1}{2}e^2 - \frac{3}{2}.$$

Notice that if we tried to do it in the other order, it would be tough going: $\int ye^{xy}\,dy$ would require an integration by parts. In some cases, it's even worse: the integral simply can't be done unless the order is right.

b) *What is the average value of $f(x, y) = ye^{xy}$ on the region R?*
It's given by

$$\frac{1}{\text{area}(R)} \int_0^2 \int_0^1 ye^{xy}\,dy\,dx = \frac{1}{2}\left(\frac{1}{2}e^2 - \frac{3}{2}\right) = \frac{1}{4}e^2 - \frac{3}{4}.$$

Problem 3. a) *Sketch the region of integration for*

$$\int_0^2 \int_0^{x^2} y\,dy\,dx.$$

It's the region between $y = 0$ and $y = x^2$ with $0 \le x \le 2$.

b) *Evaluate the integral.*
Inner:

$$\int_0^{x^2} y\,dy = \frac{y^2}{2}\Big|_0^{x^2} = \frac{x^4}{2}.$$

Outer:

$$\int_0^2 \frac{x^4}{2}\,dx = \frac{x^5}{10}\Big|_0^2 = \frac{32}{10} = \frac{16}{5}.$$

c) *Rewrite the integral with the variables in the opposite order.*
This time y is going to go on the outside. Based on the picture, we need to go from $y = 0$ to $y = 4$. For a given value of y, what x's do we want? It's frmo $x = 0$ to $\sqrt{y}$ to 2.

$$\int_0^4 \int_{\sqrt{y}}^2 y\,dx\,dy.$$

d) *Evaluate the integral.*
Inner:

$$\int_{\sqrt{y}}^2 y\,dx = y(2 - \sqrt{y}) = 2y - y^{3/2}.$$

Outer:

$$\int_0^4 2y - y^{3/2}\,dy = \left(y^2 - \frac{2}{5}y^{5/2}\right)\Big|_0^4 = 16 - \frac{2}{5}32 = \frac{16}{5}.$$