Math 210 (Lesieutre)
13.3: Double integrals in other coordinate systems
March 3, 2017

**Problem 1.** *Evaluate the integral*

$$\int_0^1 \int_{x^2}^1 xy\,dy\,dx$$

Inner:

$$\int_{x^2}^1 xy\,dy = x\frac{y^2}{2}\Big|_{x^2}^1 = \frac{x}{2} - \frac{x^5}{2}.$$

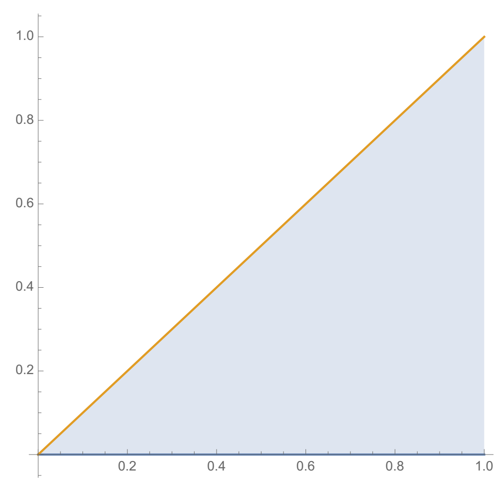
Outer:

$$\int_0^1 \left(\frac{x}{2} - \frac{x^5}{2}\right)\,dx = \left(\frac{x^2}{4} - \frac{x^6}{12}\right)\Big|_0^1 = \left(\frac{1}{4} - \frac{1}{12}\right) - (0,0) = \frac{1}{6}.$$

**Problem 2.** *Sketch the region of integration for each of the following double integrals.*

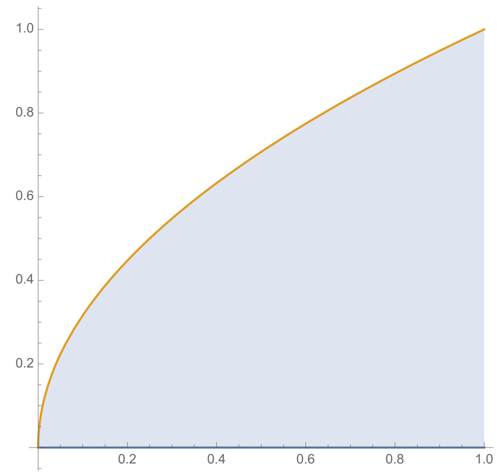
a) $\int_0^1 \int_0^x f(x,y)\,dy\,dx$

Here $x$ goes from 0 to 1, and for a given value of $x$, $y$ goes from 0 to $x$. This makes a triangle.



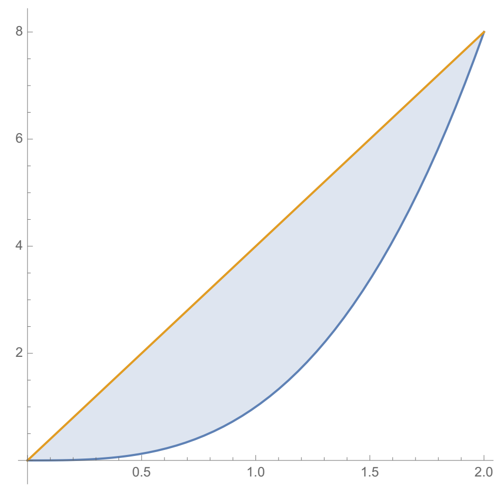

b) $\int_0^1 \int_{y^2}^1 f(x,y)\,dx\,dy$

Here $y$ goes from 0 to 1, and for a given value of $y$, $x$ goes from $y^2$ to 1. Remembering that $x = y^2$ is a sideways parabola, here's the picture we get:




c) $\int_0^2 \int_{x^3}^{4x} f(x,y)\,dy\,dx$

Here $x$ goes from 0 to 2, and for a given value of $x$, $y$ goes from $x^3$ to $4x$. Here it is:




**Problem 3.** *For each of the double integrals in the previous problem, write bounds in new coordinates to reverse the order of integration.*

a) $\int_0^1 \int_0^x f(x,y)\,dy\,dx$

Based on the picture, the new bounds are

$$\int_0^1 \int_y^1 f(x,y)\,dx\,dy$$

b) $\int_0^1 \int_{y^2}^1 dx\,dy$

Again, go to the picture. $x$ is going to run from 0 to 1, and for a given $x$, $y$ goes from 0 to $\sqrt{x}$.

$$\int_0^1 \int_0^{\sqrt{x}} f(x,y)\, dy\, dx.$$

c) $\displaystyle\int_0^2 \int_{x^3}^{4x} dy\, dx$

Now we need to go in the other direction. We can see that $y$ goes from 0 to 8. What are the bounds for a given value of $y$? The lower bound is the orange curve, which is $y = 4x$, aka $x = y/4$. The upper bound is the blue curve, $y = x^3$, so $x = \sqrt[3]{y}$. Hence the integral is

$$\int_0^8 \int_{y/4}^{\sqrt[3]{y}} f(x,y)\, dx\, dy.$$

**Problem 4.** *Find the area of the region in (b) by integrating the function $f(x,y) = 1$. Check that you get the same answer for either order of integration.*

First, we want to evaluate

$$\int_0^1 \int_{y^2}^1 1\, dx\, dy.$$

Inner:

$$\int_{y^2}^1 1\, dx = x\Big|_{y^2}^1 = 1 - y^2$$

Outer:

$$\int_0^1 1 - y^2\, dy = \left(y - \frac{y^3}{3}\right)\Big|_0^1 = \frac{2}{3}.$$

Let's try to do the same thing the other way:

$$\int_0^1 \int_0^{\sqrt{x}} 1\, dy\, dx.$$

Inner:

$$\int_0^{\sqrt{x}} 1\, dy = y\Big|_0^{\sqrt{x}} = \sqrt{x}.$$

Outer:

$$\int_0^1 \sqrt{x}\, dx = \frac{2}{3}x^{3/2}\Big|_0^1 = \frac{2}{3}.$$

That's the same answer we got before, as it should be.